

ROOT SYSTEMS AND DIAGRAM CALCULUS.

III. SEMI-COXETER ORBITS OF LINKAGE DIAGRAMS AND THE CARTER THEOREM

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ABSTRACT. A diagram obtained from the Carter diagram Γ by adding one root together with its bonds such that the resulting subset of roots is linearly independent is said to be the *linkage diagram*. Given a linkage diagram, we associate the linkage labels vector, which is introduced like the vector of Dynkin labels. Similarly to the dual Weyl group, we introduce the group W_L^\vee associated with Γ , and we call it the dual partial Weyl group. The linkage labels vectors connected under the action of W_L^\vee constitute the linkage system $\mathcal{L}(\Gamma)$, which is similar to the weight system arising in the representation theory of the semisimple Lie algebras. The Carter theorem states that every element of a Weyl group W is expressible as the product of two involutions. We give the proof of this theorem based on the description of the linkage system $\mathcal{L}(\Gamma)$ and semi-Coxeter orbits of linkage labels vectors for any Carter diagram Γ . The main idea of the proof is based on the fact that, with a few exceptions, in each semi-Coxeter orbit there is a special linkage diagram – called *unicolored*, for which the decomposition into the product of two involutions is trivial.

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Gelfand requested that I review the H. Weyl - Van der Waerden papers on semisimple Lie groups. I found them very difficult to read, and I tried to find my own ways. It came to my mind that there is a natural way to select a set of generators for a semisimple Lie algebra by using simple roots (i.e., roots which cannot be represented as a sum of two positive roots). Since the angle between any two simple roots can be equal only to $\pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$, a system of simple roots can be represented by a simple diagram. An article was submitted to *Matematicheskii Sbornik* in October 1944, [Dy46]. Only a few years later, when recent literature from the West reached Moscow, I discovered that similar diagrams have been used by Coxeter for describing crystallographic groups.

E. B. Dynkin, Foreword in "Selected papers of E. B. Dynkin with commentary", [Dy00, p. 2]

1. Introduction

1.1. The Carter theorem. In the present paper we give the proof of the Carter theorem on the decomposition of every element of any Weyl group W into the product of two involutions.

Theorem 1.1 ([Ca72], Theorem C). *The following equivalent statements hold for the Weyl group:*

- (i) *Every element of a Weyl group W is expressible as the product of two involutions.*
- (ii) *Every element of W is contained in some dihedral subgroup.*
- (iii) *For each element $w \in W$ there is an involution $i \in W$ such that $iwi = w^{-1}$.*

Corollary 1.2. *Every element of W is conjugate to its inverse.*

In [Sp74], Springer gives a proof of the Carter theorem for all finite Coxeter groups including the non-crystallographic cases I_n (dihedral group), H_3 and H_4 . Springer deduced the proof from the classification of so-called regular elements in the Coxeter groups and by inspection from the known character tables of the irreducible Weyl groups, [Sp74, §8.6, §8.7].

The proof given by Carter in [Ca72] uses the calculation of all conjugacy classes in the Weyl group. Our proof uses the classification of linkage systems and semi-Coxeter orbits for every Carter diagrams. The definitions of linkage systems and semi-Coxeter orbits will be given below in Section 1.2. The linkage systems for Carter diagrams from $C4 \amalg DE4$ are presented in [St10.II], where $C4$ is the class of Carter diagrams, each of which contains 4-cycle $D_4(a_1)$ as a subdiagram, and $DE4$ is the class of Carter diagrams, each of which contains D_4 as a subdiagram, see Section 1.2.4. In this paper, we give the complete description of semi-Coxeter orbits for any $\Gamma \in C4 \amalg DE4$. The linkage systems and semi-Coxeter orbits for A_l and B_l will be presented in [St11].

1.2. The Carter diagrams and semi-Coxeter elements.

1.2.1. Solid and dotted edges. The Carter diagram (= admissible diagram) [Ca72, §4] is the diagram Γ satisfying two conditions:

- (a) The nodes of Γ correspond to a set of linearly independent roots.
- (b) Each subgraph of Γ which is a cycle contains even number of vertices.

Let $w = w_1 w_2$ be the decomposition of w into the product of two involutions. By [Ca72, Lemma 5] each of w_1 and w_2 can be expressed as a product of reflections corresponding to mutually

orthogonal roots:

$$w = w_1 w_2, \quad w_1 = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}, \quad w_2 = s_{\beta_1} s_{\beta_2} \dots s_{\beta_h}, \quad \text{where } k + h = l_C(w). \quad (1.1)$$

For details, see [Ca72, §4], [St10.I, §1.1]. We denote by α -set (resp. β -set) the subset of roots corresponding to w_1 (resp. w_2):

$$\alpha\text{-set} = \{\alpha_1, \alpha_2, \dots, \alpha_k\}, \quad \beta\text{-set} = \{\beta_1, \beta_2, \dots, \beta_h\}. \quad (1.2)$$

Any coordinate from α -set (resp. β -set) of the linkage labels vector we call α -label (resp. β -label). The decomposition (1.1) is said to be the *bicolored decomposition*.

For the Dynkin diagrams, a number of bonds for non-orthogonal roots describes the angle between roots, and the ratio of lengths of two roots. For the Carter diagrams, we add designation distinguishing acute and obtuse angles between roots. Recall, that for the Dynkin diagrams, all angles between simple roots are obtuse and a special designation is not necessary. A *solid edge* indicates an obtuse angle between roots exactly as for simple roots in the case of Dynkin diagrams. A *dotted edge* indicates an acute angle between the roots considered, see [St10.I]. For examples of diagrams with dotted and solid edges, see Table 2.2.

1.2.2. Semi-Coxeter elements. A conjugacy class of W which can be described by a connected Carter diagram with number of nodes equal to the rank of W is called a *semi-Coxeter* class, [CE72] (or, a *primitive* conjugacy class, [KP85]). Let us fix some basis of roots corresponding to the given Carter diagram Γ :

$$\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_h\}, \quad (1.3)$$

where α_i, β_j are roots (not necessarily simple) corresponding to Γ . The element

$$\mathbf{c} = w_\alpha w_\beta, \quad \text{where } w_\alpha = \prod_{i=1}^k s_{\alpha_i}, \quad w_\beta = \prod_{j=1}^h s_{\beta_j}, \quad (1.4)$$

given in the basis (1.3) we call the semi-Coxeter element. It is the representative of the semi-Coxeter class. The dual semi-Coxeter element

$$\mathbf{c}^* = {}^t w_\alpha {}^t w_\beta, \quad \text{where } w_\alpha = \prod_{i=1}^k s_{\alpha_i}^*, \quad w_\beta = \prod_{j=1}^h s_{\beta_j}^*, \quad (1.5)$$

is used for the proof of the Carter theorem. Semi-Coxeter elements for diagrams $D_l, D_l(a_k), E_l, E_l(a_k)$, where $l \leq 7$, are presented in Tables A.3-A.5.

Note that roots (1.3) are not necessarily simple. If all roots (1.3) are simple, the Carter diagram Γ is a Dynkin diagram and the semi-Coxeter element (1.4) coincides with the corresponding Coxeter element.

1.2.3. Linkages, linkage diagrams and linkage systems. Let $w = w_1 w_2$ be the bicolored decomposition of some element $w \in W$, where w_1, w_2 are two involutions, associated, respectively, with α -set $\{\alpha_1, \dots, \alpha_k\}$ and β -set $\{\beta_1, \dots, \beta_h\}$ of roots from the root system Φ , see (1.1), (1.2), and Γ be the Carter diagram associated with this bicolored decomposition. We consider the *extension* of the root basis Π_w by means of the root $\gamma \in \Phi$, such that the set of roots

$$\Pi_w(\gamma) = \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_h, \gamma\} \quad (1.6)$$

is linearly independent. Let us multiply w on the right by the reflection s_γ corresponding to γ and consider the diagram $\Gamma' = \Gamma \cup \gamma$ together with new edges. These edges are

$$\begin{cases} \text{solid, for } (\gamma, \tau) = -1, \\ \text{dotted, for } (\gamma, \tau) = 1, \end{cases}$$

where τ one of elements (1.6). The diagram Γ' is said to be the *linkage diagram*, and the root γ is said to be the *linkage* or the γ -*linkage*. Consider vectors γ^\vee belonging to the dual space L^\vee and defined by (1.7). Vector (1.7) is said to be *linkage labels vector* or, for brevity, *linkage labels*. There is, clearly, the one-to-one correspondence between linkage labels vectors γ^\vee (with labels $\gamma_i^\vee \in \{0, -1, 1\}$) and simply-laced linkage diagrams (i.e., such linkage diagrams that $(\gamma, \tau) \in \{0, -1, 1\}$).

$$\gamma^\vee := \begin{pmatrix} (\gamma, \alpha_1) \\ \vdots \\ (\gamma, \alpha_k) \\ (\gamma, \beta_1) \\ \vdots \\ (\gamma, \beta_h) \end{pmatrix} \quad (1.7)$$

Let L be the linear space spanned by the roots associated with Γ , The linkage labels vector is the element of the dual linear space L^\vee . We denote the linkage labels vector by γ^\vee . A certain group W_L^\vee named the dual partial Weyl group acts in the dual space L^\vee . This group acts on the linkage label vectors, i.e., on the set of linkage diagrams:

$$(w\gamma)^\vee = w^* \gamma^\vee,$$

where $w^* \in W_L^\vee$, see Proposition 2.9 from [St10.II]. The set of linkage diagrams (=linkage labels) under action of W_L^\vee constitute the diagram called the *linkage system* similarly to the weight system in the theory of representations of semisimple Lie algebras, see [Sl81, p. 30], [St10.II, p. 4].

Remark 1.3. By abuse of language, we sometimes say *linkages* instead of *linkage diagrams*. Similarly, remembering only the algebraic nature of the linkage diagram, we use the term *linkage label vector* or *linkage labels*.

1.2.4. *Classes of Carter diagrams.* We divide all Carter diagrams to the following classes:

Simply-laced Carter diagrams:

1. DE4, Dynkin diagrams containing D_4 as a subdiagram,
2. C4, Carter diagrams containing 4-cycle $D_4(a_1)$ as a subdiagram,
3. A, Dynkin diagrams A_l ,

Multiply-laced Carter diagrams:

4. BC, Dynkin diagrams B_l, C_l ,
5. FG, Dynkin diagrams F_4, G_2 , and the 4-cycle with two double bonds $F_4(a_1)$.

For $\Gamma \in \text{C4} \amalg \text{DE4}$, the linkage systems are described in [St10.II]. In this case, for $l \leq 7$, the linkage systems $\mathcal{L}(\Gamma)$ looks as follows: every linkage diagram containing at least one non-zero α -label (see Section 1.2.1) belongs to a certain 8-cell "spindle-like" linkage subsystem called *loctet* (= linkage octet). The loctets are the main construction blocks in every linkage system. If all α -labels (resp. β -labels) of γ^\vee are zeros, the linkage diagram γ^\vee is said to be β -*unicolored* (resp. α -*unicolored*) linkage diagram. Every linkage system is the union of several loctets and several β -unicolored linkage diagrams, see [St10.II, §3]. In the case, where $l > 7$, the linkage systems for two infinite series D_l and $D_l(a_k)$ are described as follows: for $D_l(a_k)$ the linkage system looks as *wind rose of linkages*, see [St10.II, Fig. B.46-B.47]; for D_l the linkage system looks as the Carter diagram $D_l(a_k)$, see [St10.II, Fig. B.48].

Remark 1.4 (multiply-laced Carter diagrams FG). For any $\Gamma \in \text{FG}$, the linkage system is trivial. Really, for the case G_2 , there are maximum two linearly independent roots. Thus, the linkage system $\mathcal{L}(G_2)$ is trivial, see [St10.I, Rem. 2.2]. Further, for the multiply-laced 4-cycle $F_4(a_1)$, there is no additional fifth edge, otherwise such a diagram contains an extended Dynkin diagram as a subdiagram, see [St10.I, §A.3.2]. The simple extension of F_4 leads to the subdiagram, which

is one of extended Dynkin diagrams \widetilde{F}_{41} , \widetilde{F}_{42} , see [St10.I, Example A.3], or \widetilde{CD}_n , \widetilde{DD}_n that can not be. Any triangle extending F_4 is also moved to one of cases \widetilde{F}_{41} , \widetilde{F}_{42} , see [St10.II, §4.3]. \square

In [St11], we will construct remaining cases of linkage systems for two infinite series A_l and B_l .

2. The proof of the Carter theorem

2.1. Linear independency and reduced decomposition.

2.1.1. *Reduced decomposition and the Carter length $l_C(w)$.* Each element $w \in W$ can be expressed in the form

$$w = s_{\tau_1} s_{\tau_2} \dots s_{\tau_k}, \quad \tau_i \in \Phi, \quad (2.1)$$

where Φ is the root system associated with the Weyl group W ; s_{τ_i} are reflections in W corresponding to not necessarily simple roots $\tau_i \in \Phi$. We denote by $l_C(w)$ the smallest value k in any expression like (2.1). The Carter length $l_C(w)$ is always less than the classical length $l(w)$. The decomposition (2.1) is called *reduced* if $l_C(s_{\tau_1} s_{\tau_2} \dots s_{\tau_k}) = k$, i.e., the number of reflections in (2.1) can not be decreased.

Lemma 2.1. [Ca72, Lemma 3] *Let $\tau_1, \tau_2, \dots, \tau_k \in \Phi$. Then $s_{\tau_1} s_{\tau_2} \dots s_{\tau_k}$ is reduced if and only if $\tau_1, \tau_2, \dots, \tau_k$ are linearly independent.*

\square

2.1.2. *The basic conjugacy relation.*

Lemma 2.2 (on conjugacy). *Let $\{\tau_1, \dots, \tau_n\}$ be the subset of linearly independent roots (not necessarily simple), $\tau_i \in \Phi$, and let $w \in W$ be the element, which is decomposed into the product of reflections $\{s_{\tau_1}, \dots, s_{\tau_n}\}$.*

1) *If γ such a root that $\{\gamma, \tau_1, \dots, \tau_n\}$ are linearly independent then $\{w\gamma, \tau_1, \dots, \tau_n\}$ are also linearly independent.*

2) *The following conjugacy relation holds for any integer k :*

$$s_\gamma w \simeq s_{w^k \gamma} w.$$

In particular, for the semi-Coxeter element \mathbf{c} , we have

$$s_\gamma \mathbf{c} \simeq s_{\mathbf{c}^k \gamma} \mathbf{c}. \quad (2.2)$$

Proof. 1) Since $s_{\tau_i}(\tau_j) \in \{\tau_i, \tau_j\}$, and $s_{\tau_i}(\gamma) \in \{\gamma, \tau_i\}$, we have

$$w\{\tau_1, \dots, \tau_n\} \subseteq \{\tau_1, \dots, \tau_n\}, \text{ and}$$

$$w\gamma \in \{\gamma, \tau_1, \dots, \tau_n\}, \text{ i.e., } w\gamma = \gamma + \sum_{i=1}^n a_i \tau_i$$

for some rational factors a_i . If $\{w\gamma, \tau_1, \dots, \tau_n\}$ are linearly dependent, we have

$$w\gamma = \sum_{i=1}^n b_i \tau_i \text{ for some rational } b_i, \text{ i.e., } \gamma + \sum_{i=1}^n a_i \tau_i = \sum_{i=1}^n b_i \tau_i$$

that contradicts to the linear independency of $\{\gamma, \tau_1, \dots, \tau_n\}$.

2) Let

$$w = \prod_{i=1}^m s_{\tau_i}, \quad (2.3)$$

where not necessarily all τ_i are different. For example, it can be that $m > n$. We have

$$\begin{aligned} s_\gamma w &= s_\gamma s_{\tau_1} \dots s_{\tau_m} = s_{\tau_1} s_{s_{\tau_1}(\gamma)} s_{\tau_2} \dots s_{\tau_m} = s_{\tau_1} s_{\tau_2} s_{s_{\tau_2} s_{\tau_1}(\gamma)} s_{\tau_3} \dots s_{\tau_m} = \dots = \\ &= s_{w s_{w^{-1}} \gamma} \simeq s_{w^{-1} \gamma} w \simeq \dots \simeq s_{w^{-k} \gamma} w, \text{ for } k > 0. \end{aligned} \quad (2.4)$$

By mapping $\gamma' = w^{-k} \gamma$ we obtain from (2.4) also that

$$s_{\gamma'} w \simeq s_{w^k \gamma'}$$

for any integer $k > 0$ and every root γ' . Thus, (2.2) is proven. \square

Eq. (2.2) is the basic relation in our proof of the Carter theorem, see Section 2.2.1.

2.2. The proof of Theorem 1.1.

2.2.1. The induction step. Let \mathbf{c} be the semi-Coxeter element associated with the Carter diagram Γ such that \mathbf{c} has bicolored decomposition given by (1.4). The proof of the theorem is carried out by induction on the Carter length $l_C(\mathbf{c})$ of the decomposition (1.4), see Section 2.1.1. For details, see [Ca72], [St10.I, p. 4]. Suppose, γ is the root such that roots $\{\gamma, \alpha_1, \dots, \beta_h\}$ are linearly independent. According to Lemma 2.2, heading 1) $\{\mathbf{c}^n \gamma, \alpha_1, \dots, \beta_h\}$ are also linearly independent.

We will show that

$$s_\gamma \mathbf{c} \text{ is also associated with a certain Carter diagram.} \quad (2.5)$$

Of course, it suffices to prove the property (2.5) for any conjugate of $s_\gamma \mathbf{c}$. The property (2.5) gives us the induction step. According to Lemma 2.2,2) it suffices to find such an integer n that any conjugate of $s_{\mathbf{c}^n \gamma} \mathbf{c}$ has the bicolored decomposition.

2.2.2. Semi-Coxeter orbits. We have $(\mathbf{c}^n \gamma)^\vee = (\mathbf{c}^n)^* \gamma^\vee = (\mathbf{c}^*)^n \gamma^\vee$ for any n , see [St10.II, Proposition 2.9]. Let us consider the sequence of linkages

$$(\mathbf{c}^n \gamma)^\vee = (\mathbf{c}^*)^n \gamma^\vee, \quad n = 0, \pm 1, \pm 2, \dots \quad (2.6)$$

It is clear that (2.6) is the finite periodic sequence, see Tables A.3-A.5. This sequence is said to be the *semi-Coxeter orbit*. Remember, that the linkage diagram γ^\vee is said to be the α -unicolored (resp. β -unicolored) linkage diagram if all β -labels (resp. α -labels), i.e., coordinates corresponding to all β_i (resp. α_i) of γ^\vee are zeros, [St10.II, p. 5]. Suppose, for some integer m , the element $(\mathbf{c}^*)^m \gamma^\vee$ in semi-Coxeter orbit is a certain unicolored linkage diagram. Let $(\mathbf{c}^*)^m \gamma^\vee$ be, for example, α -unicolored. Then $(\mathbf{c}^m \gamma, \beta_i) = 0$ for all β -labels. This means that $s_{\mathbf{c}^m \gamma}$ commute with all s_{β_i} . By (2.2) and (1.4)

$$s_\gamma \mathbf{c} = s_{\mathbf{c}^m \gamma} \mathbf{c} = s_{\mathbf{c}^m \gamma} \prod_{i=1}^k s_{\alpha_i} \prod_{j=1}^h s_{\beta_j} \simeq \prod_{i=1}^k s_{\alpha_i} \left(\prod_{j=1}^h s_{\beta_j} \right) s_{\mathbf{c}^m \gamma}.$$

The latter product is the bicolored decomposition, since $(\prod_{j=1}^h s_{\beta_j}) s_{\mathbf{c}^m \gamma}$ is involution. Thus, it suffices to prove that any semi-Coxeter orbit contains an unicolored linkage diagram.

2.2.3. Unicolored linkage diagrams and exceptional orbits. However, there are semi-Coxeter orbits containing no unicolored linkage diagrams. We call these orbits *exceptional semi-Coxeter orbits*. The total quantity of orbits for Carter diagrams from $\Gamma \in \mathbf{C4} \amalg \mathbf{DE4}$ (for $l \leq 7$) is 140, the number of exceptional orbits is 24, see Table 2.1. Instead 24 orbits it suffices to consider only 10 exceptional orbits, namely $(1a)$, $(2a)$, $(2c)$, $(3a)$, $(4a)$, $(4b)$, $(5a)$, $(6a)$, $(7a)$, $(7b)$, see Table 2.2, they are checked case-by-case in Section 2.3.

The Carter diagram	Number of orbits			Lengths of orbits ¹	Number of linkages
	All	no unicolored linkages	self-opposite		
$D_4(a_1)$	6	2	6	6×4	24
D_4	6	-	6	$3 \times 6 + 3 \times 2$	24
$D_5(a_1)$	6	-	2	$2 \times 12 + 3 \times 4 + 6$	42
D_5	6	-	2	$5 \times 8 + 2$	42
$E_6(a_1)$	6	-	-	9×9	54
$E_6(a_2)$	10	4	-	$8 \times 6 + 2 \times 3$	54
E_6	6	-	-	$4 \times 12 + 2 \times 3$	54
$D_6(a_1)$	10	4	2	$9 \times 8 + 4$	76
$D_6(a_2)$	14	2	14	$12 \times 6 + 2 \times 2$	76
D_6	8	-	8	$7 \times 10 + 3 \times 2$	76
$E_7(a_1)$	4	-	4	4×14	56
$E_7(a_2)$	6	2	2	$4 \times 12 + 6 + 2$	56
$E_7(a_3)$	4	-	4	$30 + 2 \times 10 + 6$	56
$E_7(a_4)$	10	6	10	$9 \times 6 + 2$	56
E_7	4	-	4	$3 \times 18 + 2$	56
$D_7(a_1)$	10	-	2	$6 \times 20 + 2 \times 4 + 10 + 4$	142
$D_7(a_2)$	10	-	2	$4 \times 24 + 4 \times 8 + 8 + 6$	142
D_7	14	4	2	$10 \times 12 + 2 \times 4 + 12 + 2$	142
$D_l(a_k), l > 7$	2	-	2	$2 \times (k + 1) + 2 \times (l - k - 1)$	$2l$
$D_l, l > 7$	2	-	2	$2 \times (l - 1)$	$2l$

TABLE 2.1. Number and lengths of semi-Coxeter orbits

Remark 2.3. We observe that the number of unicolored linkage diagrams in every semi-Coxeter orbit is equal to 0 (exceptional orbit) or 2, see Tables B.6 - B.19 (where unicolored linkage diagrams are framed by a rectangle). Of course, this fact requires *a priori* reasoning.

¹Explanation to the column. For example, expression $6 \times 20 + 2 \times 4 + 10 + 4$ in the line $D_7(a_1)$ means that the total number of linkage diagrams in the linkage system for the Carter diagram $D_7(a_1)$ is divided into the sum of 6 orbits each of 20 elements, 2 orbits each of 4 elements, one orbit containing 10 elements and one orbit containing 4 elements.

2.2.4. *Semi-Coxeter orbits for infinite series $D_l(a_k)$ and D_l .* It is convenient to imagine a semi-Coxeter orbit for D_l (resp. $D_l(a_k)$) as a cosine wave that runs in one direction and then returns in the opposite direction with a shift in the phase by half a period, see Fig. 2.1 and Fig. 2.2.

Diagram D_l . We have one long orbit – the red wave in the horizontal direction, and one 2-element orbit – the blue orbit in the vertical direction, see Fig. 2.1. There are two cases: $l = 2p + 2$ and $l = 2p + 1$. For $l = 2p + 2$, the linkage labels vector $\gamma_{\alpha_p^+}^\vee$ (resp. $\gamma_{\alpha_p^-}^\vee$) is the vector with the unit in the place α_p^+ (resp. α_p^-) and zeros in remaining places, see Fig. 2.1. They are two unicolored linkages for the long semi-Coxeter orbit (red wave). The blue orbit consists of following two unicolored linkages: $\gamma_4^\vee = \{0, 1, -1, 0, \dots, 0\}$, $\gamma_5^\vee = -\gamma_4^\vee$ with the only non-zero coordinates in coordinates α_2 and α_3 . Notations of γ_4^\vee , γ_5^\vee are retained as in [St10.II, Fig. B.48]. For $l = 2p + 1$, the linkage labels vector $\gamma_{\beta_p^+}^\vee$ (resp. $\gamma_{\beta_p^-}^\vee$) is the vector with the unit in the place β_p^+ (resp. β_p^-) and zeros in remaining places, see Fig. 2.1. The blue orbit is the same as in the case $l = 2p + 2$. Linkages $\gamma_{\alpha_p^+}^\vee$ and $\gamma_{\alpha_p^-}^\vee$ (see Remark 1.3) for $l = 2p + 2$, and linkages $\gamma_{\beta_p^+}^\vee$ and $\gamma_{\beta_p^-}^\vee$ for $l = 2p + 1$ are the same linkages as $\gamma_{\tau_{l-3}^+}^\vee$ and $\gamma_{\tau_{l-3}^-}^\vee$ in [St10.II, Fig. B.48].

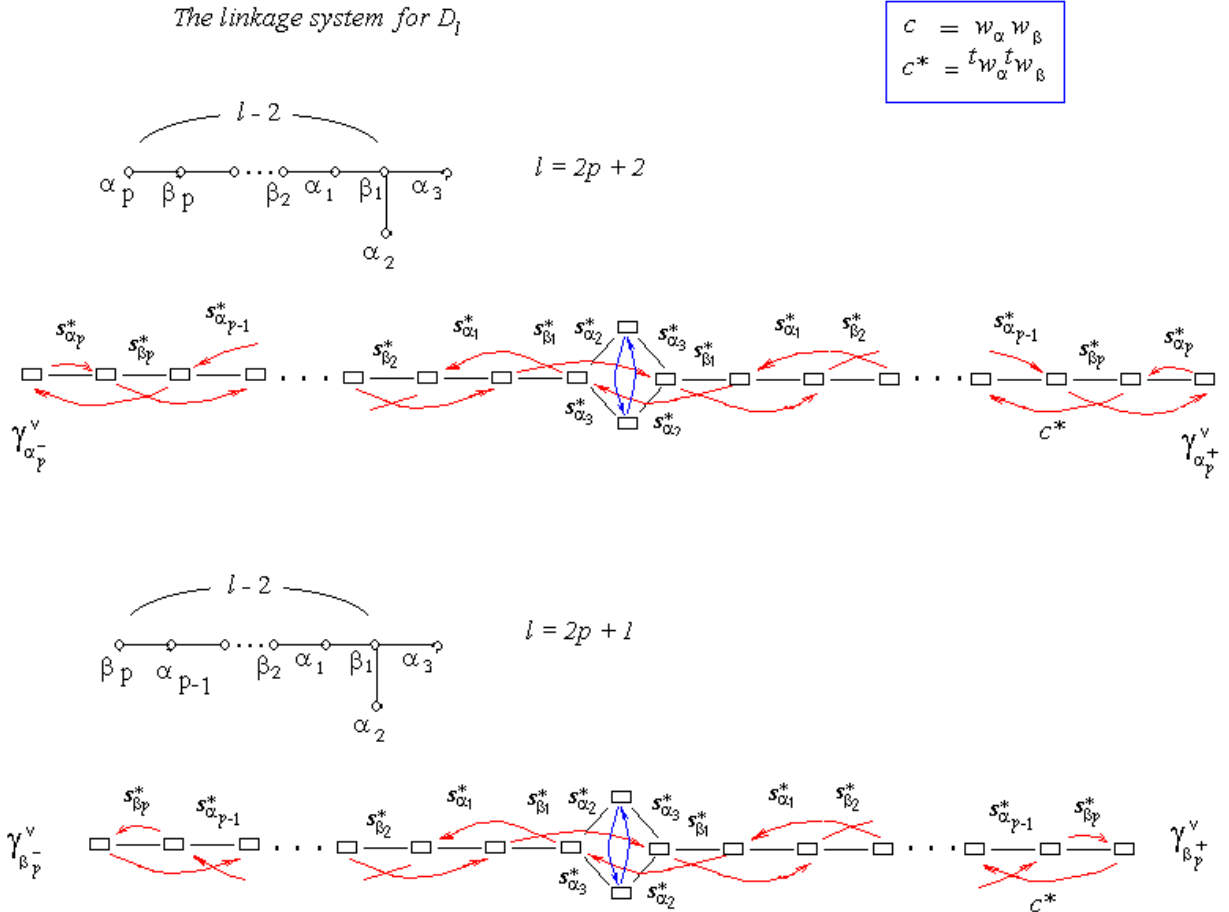


FIGURE 2.1. Two semi-Coxeter orbits of D_l , one of length $2(l - 1)$ (= Coxeter number), one of length 2

Diagram $D_l(a_k)$. Here, we have two long orbits: one red wave in the horizontal direction, and one blue wave in the vertical direction, see Fig. 2.2. The linkage labels vector $\gamma_{\tau_{k-1}}^+$ (resp. $\gamma_{\tau_{k-1}}^-$, $\gamma_{\varphi_{l-k-2}}^+$, $\gamma_{\varphi_{l-k-2}}^-$) is the vector with the unit on the place τ_{k-1}^+ (resp. τ_{k-1}^- , φ_{l-k-2}^+ , φ_{l-k-2}^-) and zeros on remaining places, see Fig. 2.2. As above, these vectors are unicolored, see [St10.II, Fig. B.46-B.47].

Two semi-Coxeter orbits of $D_l(a_k)$ are of lengths $2(k+1)$ and $2(l-k-1)$. For the left (resp. right) branch of $D_l(a_k)$, there are two options for endpoints: α_p or β_p (resp. α_q or β_q). Thus, from the view of endpoints there are 4 options for the Carter diagram $D_l(a_k)$:

$$\{\alpha_p, \alpha_q\}, \quad \{\alpha_p, \beta_q\}, \quad \{\beta_p, \alpha_q\}, \quad \{\beta_p, \beta_q\}.$$

In Fig. 2.2 we depict only one from 4 options for $D_l(a_k)$ and its linkage system $\mathcal{L}(D_l(a_k))$.

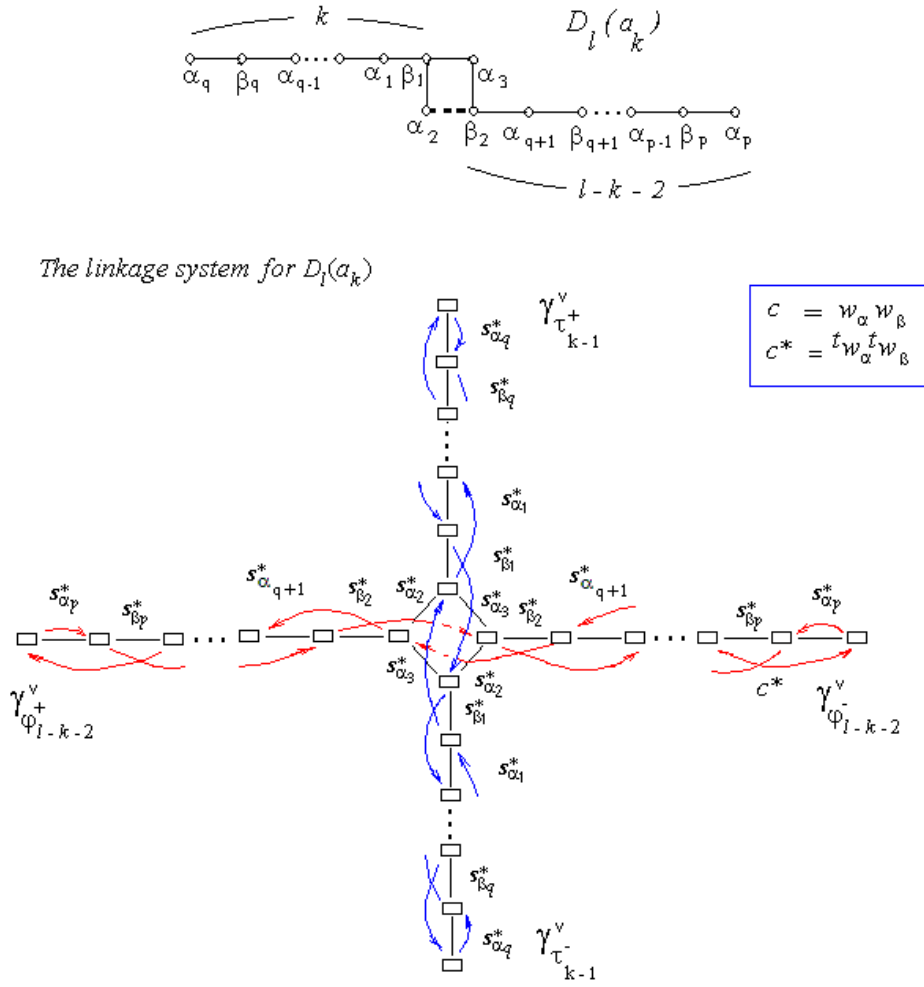


FIGURE 2.2. Two semi-Coxeter orbits of D_l : one of length $2(k+1)$, one of length $2(l-k-1)$

2.3. Exceptional semi-Coxeter orbits.


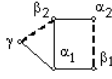
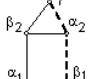
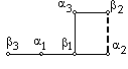

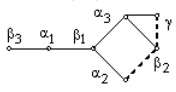
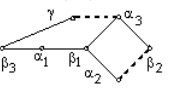
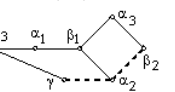
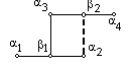


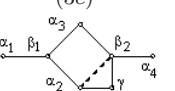
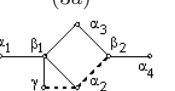
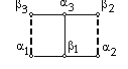
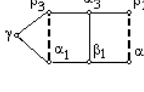
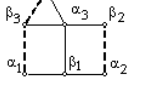
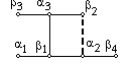
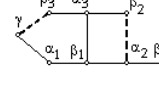
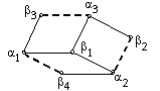
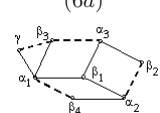
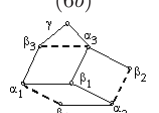
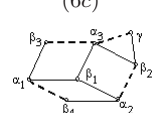
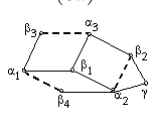
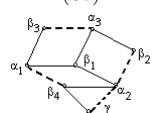
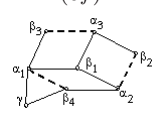
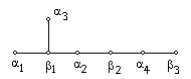
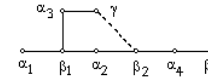
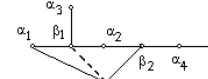
	The Carter diagram Γ	Total orbits	Representatives of exceptional orbits in the linkage system $\mathcal{L}(\Gamma)$
1	 $D_4(a_1)$	6	<div style="display: flex; justify-content: space-around;"> <div style="text-align: center;">  (1a) </div> <div style="text-align: center;">  (1b) </div> </div> <div style="display: flex; justify-content: space-around;"> $\{-1, 0, 0, 1\}$ $\{0, 1, 0, -1\}$ </div>
2	 $D_6(a_1)$	10	<div style="display: flex; justify-content: space-around;"> <div style="text-align: center;">  (2a) </div> <div style="text-align: center;">  (2b) </div> <div style="text-align: center;">  (2c) </div> <div style="text-align: center;">  (2d) </div> </div> <div style="display: flex; justify-content: space-around;"> $\{0, -1, 0, 0, -1, 0\}$ $\{0, 0, -1, 0, 1, 0\}$ $\{0, 0, 1, 0, 0, -1\}$ $\{0, 1, 0, 0, 0, -1\}$ </div>
3	 $D_6(a_2)$	14	<div style="display: flex; justify-content: space-around;"> <div style="text-align: center;">  (3a) </div> <div style="text-align: center;">  (3b) </div> <div style="text-align: center;">  (3c) </div> <div style="text-align: center;">  (3d) </div> </div> <div style="display: flex; justify-content: space-around;"> $\{0, 0, -1, 0, 0, 1\}$ $\{0, 0, 1, 0, -1, 0\}$ $\{0, -1, 0, 0, 0, -1\}$ $\{0, 1, 0, 0, -1, 0\}$ </div>
4	 $E_6(a_2)$	10	<div style="display: flex; justify-content: space-around;"> <div style="text-align: center;">  (4a) </div> <div style="text-align: center;">  (4b) </div> </div> <div style="display: flex; justify-content: space-around;"> $\{-1, 0, 0, 0, 0, -1\}$ $\{0, 0, -1, 0, 0, 1\}$ </div>
5	 $E_7(a_2)$	6	<div style="text-align: center;">  (5a) </div> $\{-1, 0, 0, 0, 0, 1, 0\}$
6	 $E_7(a_4)$	10	<div style="display: flex; justify-content: space-around;"> <div style="text-align: center;">  (6a) </div> <div style="text-align: center;">  (6b) </div> <div style="text-align: center;">  (6c) </div> </div> <div style="display: flex; justify-content: space-around;"> $\{-1, 0, 0, 0, 0, 1, 0\}$ $\{0, 0, -1, 0, 0, -1, 0\}$ $\{0, 0, 1, 0, -1, 0, 0\}$ </div> <div style="display: flex; justify-content: space-around;"> <div style="text-align: center;">  (6d) </div> <div style="text-align: center;">  (6e) </div> <div style="text-align: center;">  (6f) </div> </div> <div style="display: flex; justify-content: space-around;"> $\{0, -1, 0, 0, -1, 0, 0\}$ $\{0, -1, 0, 0, 0, 0, 1\}$ $\{1, 0, 0, 0, 0, 0, 1\}$ </div>
7	 D_7	14	<div style="display: flex; justify-content: space-around;"> <div style="text-align: center;">  (7a) </div> <div style="text-align: center;">  (7b) </div> </div> <div style="display: flex; justify-content: space-around;"> $\{0, 0, -1, 0, 0, 1, 0\}$ $\{-1, 0, 0, 0, 1, -1, 0\}$ </div>

TABLE 2.2. Exceptional orbits

2.3.1. *Diagram $D_4(a_1)$. Case (1a).* For the Carter diagram $D_4(a_1)$, there are 2 exceptional semi-Coxeter orbits with representatives (1a) and (1b). These cases are similar, see Table 2.2. We consider only (1a).

Case (1a).

$$w = s_{\alpha_1} s_{\alpha_2} s_{\beta_1} s_{\beta_2} s_{\gamma} = s_{\alpha_2} s_{\beta_2} s_{\beta_1} s_{\beta_1 + \beta_2 + \alpha_1} s_{\gamma} \stackrel{s_{\alpha_2}}{\simeq} (s_{\beta_2} s_{\beta_1}) (s_{\beta_1 + \beta_2 + \alpha_1} s_{\gamma} s_{\alpha_2}).$$

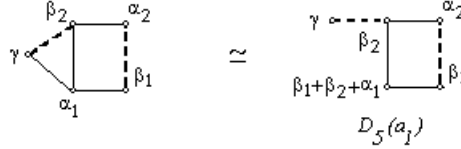


FIGURE 2.3. The linkage diagram (1a) for the Carter diagram $D_4(a_1)$

Thus, the linkage diagram (1a) from Table 2.2 is equivalent to the Carter diagram $D_5(a_1)$. This case was also considered in [St10.I, Lemma 1.8].

2.3.2. *Diagram $D_6(a_1)$. Cases (2a) and (2c).* For the diagram $D_6(a_1)$, there are 4 exceptional semi-Coxeter orbits (2a), (2b), (2c), (2d). Cases (2a) and (2b) are similar; cases (2c) and (2d) are also similar. We consider only (2a) and (2c).

Case (2a). Here, we have

$$\begin{aligned} w &= s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\gamma} = s_{\alpha_1} (s_{\alpha_2} s_{\alpha_3} s_{\beta_2}) s_{\beta_1} s_{\beta_3} s_{\gamma} = \\ & s_{\alpha_1} s_{\beta_2 + \alpha_3 - \alpha_2} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_3} s_{\gamma} \stackrel{s_{\beta_2 + \alpha_3 - \alpha_2}}{\simeq} (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}) (s_{\beta_1} s_{\beta_3} s_{\gamma} s_{\beta_2 + \alpha_3 - \alpha_2}). \end{aligned}$$

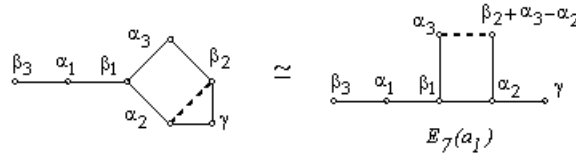


FIGURE 2.4. The linkage diagram (2a) for the Carter diagram $D_6(a_1)$

Hence, the linkage diagram (2a) from Table 2.2 is equivalent to the Carter diagram $E_7(a_1)$.

Case (2c). This case is reduced to the exception case (4a) in the exceptional orbit for $E_6(a_2)$:

$$\begin{aligned} w &= s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\gamma} = (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}) (s_{\beta_1} s_{\beta_2} s_{\beta_3 + \gamma} s_{\beta_3}) = (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}) (s_{\beta_1} s_{\beta_2} s_{-(\beta_3 + \gamma)} s_{\beta_3}) = \\ & (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}) s_{-(\beta_3 + \gamma)} (s_{\beta_1} s_{\beta_2} s_{\beta_3}) \simeq s_{-(\beta_3 + \gamma)} (s_{\beta_1} s_{\beta_2} s_{\beta_3}) (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}). \end{aligned}$$

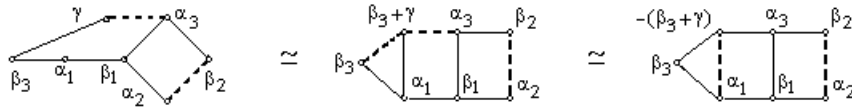


FIGURE 2.5. The linkage diagram (2c) for the Carter diagram $D_6(a_1)$

2.3.3. *Diagram $D_6(a_2)$. Case (3a).* For the Carter diagram $D_6(a_2)$, there are 4 exceptional semi-Coxeter orbits with representatives (3a), (3b), (3c), (3d). These cases are similar to each other, see Table 2.2. We consider only (3a).

Case (3a). Here, we have

$$\begin{aligned} w &= s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\beta_1} s_{\beta_2} s_{\gamma} = s_{\alpha_1} s_{\beta_2 + \alpha_3 - \alpha_2 + \alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\beta_1} s_{\gamma} = \\ & s_{\beta_2 + \alpha_3 - \alpha_2 + \alpha_4} (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}) s_{\beta_1} s_{\gamma} \stackrel{s_{\beta_2 + \alpha_3 - \alpha_2 + \alpha_4}}{\simeq} (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}) (s_{\beta_1} s_{\gamma} s_{\beta_2 + \alpha_3 - \alpha_2 + \alpha_4}) = \\ & (s_{\alpha_1} s_{-\alpha_2} s_{\alpha_3} s_{\alpha_4}) (s_{\beta_1} s_{\gamma} s_{-(\beta_2 + \alpha_3 - \alpha_2 + \alpha_4)}). \end{aligned}$$

Therefore, the linkage diagram (3a) from Table 2.2 is equivalent to the Carter diagram $E_7(a_2)$.

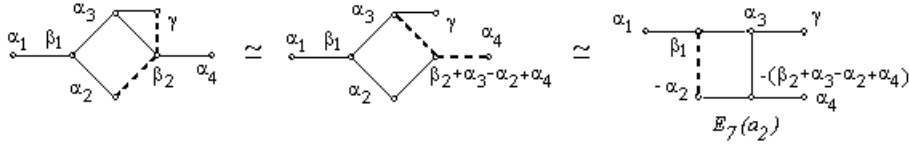


FIGURE 2.6. The linkage diagram (3a) for the Carter diagram $D_6(a_2)$

Remark 2.4 (on s -permutation). Let s_{α} and s_{β} be two adjacent reflections in the decomposition of w :

$$w = \dots s_{\alpha} s_{\beta} \dots \quad (2.7)$$

If decomposition (2.7) is written down in one of the equivalent forms

$$\begin{aligned} w &= \dots s_{\alpha} s_{\beta} \dots = \dots s_{s_{\alpha}(\beta)} s_{\alpha} \dots, \text{ or} \\ w &= \dots s_{\alpha} s_{\beta} \dots = \dots s_{\beta} s_{s_{\beta}(\alpha)} \dots, \end{aligned} \quad (2.8)$$

we say that elements s_{α} and s_{β} are s -permuted. The linkage diagram related to (2.7) is respectively changed. The corresponding transformation of the word w and related linkage diagram we call the s -permutation. In [St10.I, §1.4.1], we considered s -permutation in the framework of equivalent transformations of connection diagrams. \square

2.3.4. *Diagram $E_6(a_2)$. Cases (4a), (4b).* These two cases are different.

Case (4a). First, reflections s_{β_3} and $s_{\alpha_1} s_{\alpha_3}$ are s -permuted. The new connection between β_2 and $\beta_3 - \alpha_1 + \alpha_3$ appears, see Fig. 2.7,(b). After that, s_{β_2} and $s_{\alpha_2} s_{\alpha_3}$ are s -permuted. The the new connection disappears, see Fig. 2.7,(c).

$$w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\gamma} = s_{\alpha_2} s_{\beta_3 - \alpha_1 + \alpha_3} s_{\alpha_1} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\gamma} = s_{\beta_3 - \alpha_1 + \alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\gamma},$$

since s_{α_2} and $s_{\beta_3 - \alpha_1 + \alpha_3}$ commute, see Fig. 2.7,(b).

$$w = s_{\beta_3 - \alpha_1 + \alpha_3} s_{\alpha_1} s_{\beta_2 - \alpha_2 + \alpha_3} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\gamma} = s_{\beta_3 - \alpha_1 + \alpha_3} s_{\beta_2 - \alpha_2 + \alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\gamma},$$

since s_{α_1} and $s_{\beta_2 - \alpha_2 + \alpha_3}$ commute, see Fig. 2.7,(c). Further,

$$w = s_{\beta_3 - \alpha_1 + \alpha_3} s_{\beta_2 - \alpha_2 + \alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\gamma} = s_{-(\beta_3 - \alpha_1 + \alpha_3)} s_{-(\beta_2 - \alpha_2 + \alpha_3)} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\gamma},$$

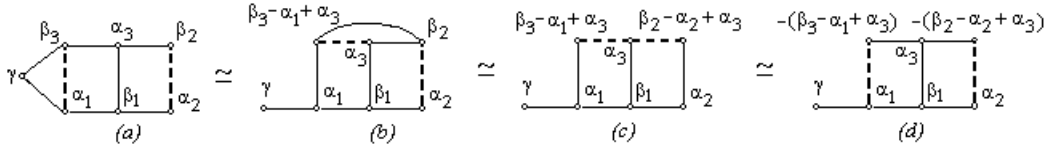
see Fig. 2.7,(d). Hence, the linkage diagram (4a) from Table 2.2 is equivalent to the Carter diagram $E_7(a_3)$.

Case (4b).

$$w = s_{\alpha_1} s_{\alpha_2} (s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\beta_3}) s_{\gamma} = s_{\alpha_1} s_{\alpha_2} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_3 + \beta_1 + \beta_2 + \beta_3} s_{\gamma},$$

where

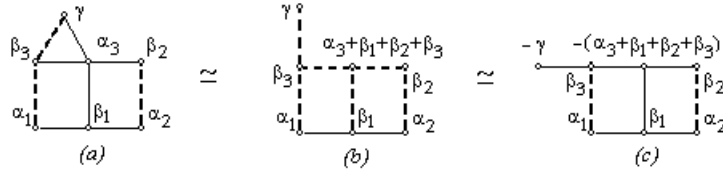
$$\begin{aligned} (\alpha_3 + \beta_1 + \beta_2 + \beta_3, \alpha_1) &= (\beta_3, \alpha_1) + (\beta_1, \alpha_1) = 0, \\ (\alpha_3 + \beta_1 + \beta_2 + \beta_3, \alpha_2) &= (\beta_2, \alpha_2) + (\beta_1, \alpha_2) = 0. \end{aligned}$$

FIGURE 2.7. The linkage diagram (4a) for the Carter diagram $E_6(a_2)$

Thus, w is described by the diagram Fig. 2.8,(b). Further,

$$w = s_{\alpha_1} s_{\alpha_2} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_3 + \beta_1 + \beta_2 + \beta_3} s_{\gamma} = s_{\alpha_1} s_{\alpha_2} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{-(\alpha_3 + \beta_1 + \beta_2 + \beta_3)} s_{\gamma},$$

see Fig. 2.8,(c). Hence, the linkage diagram (4b) from Table 2.2 is equivalent to the Carter diagram $E_7(a_3)$.

FIGURE 2.8. The linkage diagram (4b) for the Carter diagram $E_6(a_2)$

2.3.5. Diagram $E_7(a_2)$. Case (5a).

Case (5a). First, we reduce the 5-cycle to the usual contour of the 4-cycle with the adjoined triangle, see Fig. 2.9,(b).

$$w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\beta_4} (s_{\beta_3} s_{\gamma}) = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\beta_4} s_{\gamma} s_{\beta_3 - \gamma}.$$

Secondly, reflections s_{α_1} and $s_{\beta_1} s_{\beta_3 - \gamma}$ are s -permuted. The new connection between $\alpha_1 + \beta_1 - \beta_3 + \gamma$ and α_2 appears:

$$w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_2} s_{\beta_4} s_{\gamma} s_{\beta_1} s_{\beta_3 - \gamma} \stackrel{s_{\beta_1} s_{\beta_3 - \gamma}}{\simeq} (s_{\beta_1} s_{\beta_3 - \gamma} s_{\alpha_1}) s_{\alpha_2} s_{\alpha_3} s_{\beta_2} s_{\beta_4} s_{\gamma} = s_{\alpha_1 + \beta_1 - \beta_3 + \gamma} s_{\beta_1} s_{\beta_3 - \gamma} s_{\alpha_2} s_{\alpha_3} s_{\beta_2} s_{\beta_4} s_{\gamma},$$

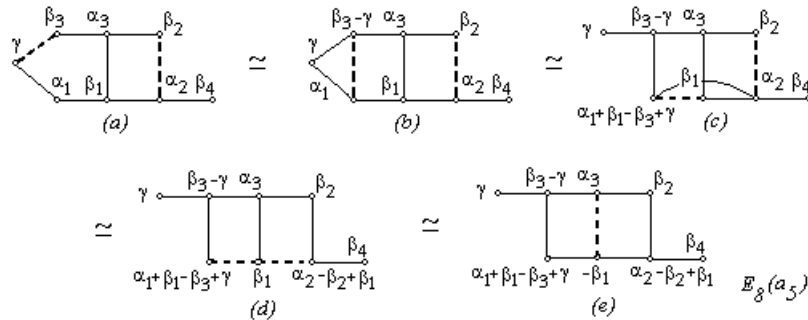
where

$$(\alpha_1 + \beta_1 - \beta_3 + \gamma, \gamma) = (\gamma, \gamma) - (\beta_3, \gamma) + (\alpha_1, \gamma) = 1 - \frac{1}{2} - \frac{1}{2} = 0,$$

$$(\alpha_1 + \beta_1 - \beta_3 + \gamma, \alpha_3) = (\beta_1 - \beta_3, \alpha_3) = 0,$$

$$(\alpha_1 + \beta_1 - \beta_3 + \gamma, \alpha_2) = (\beta_1, \alpha_2) = -\frac{1}{2},$$

see Fig. 2.9,(c).

FIGURE 2.9. The linkage diagram (5a) for the Carter diagram $E_7(a_2)$

Further, reflections α_2 and $s_{\beta_1}s_{\beta_2}$ are s -permuted. Then the new connection disappears:

$$w = s_{\alpha_1+\beta_1-\beta_3+\gamma}s_{\beta_1}s_{\beta_3-\gamma}s_{\alpha_2}s_{\alpha_3}s_{\beta_2}s_{\beta_4}s_{\gamma} \xrightarrow{s_{\beta_2}} s_{\alpha_1+\beta_1-\beta_3+\gamma}s_{\beta_3-\gamma}(s_{\beta_2}s_{\beta_1}s_{\alpha_2})s_{\alpha_3}s_{\beta_4}s_{\gamma} = s_{\alpha_1+\beta_1-\beta_3+\gamma}s_{\beta_3-\gamma}s_{\alpha_2-\beta_2+\beta_1}s_{\beta_2}s_{\beta_1}s_{\alpha_3}s_{\beta_4}s_{\gamma},$$

where

$$(\alpha_1 + \beta_1 - \beta_3 + \gamma, \alpha_2 - \beta_2 + \beta_1) = (\alpha_2, \beta_1) + (\beta_1, \beta_1) + (\beta_1, \alpha_1) = -\frac{1}{2} + 1 - \frac{1}{2} = 0.$$

Further,

$$w = s_{\alpha_1+\beta_1-\beta_3+\gamma}s_{\beta_3-\gamma}s_{\alpha_2-\beta_2+\beta_1}s_{\beta_2}s_{\beta_1}s_{\alpha_3}s_{\beta_4}s_{\gamma} = s_{\alpha_1+\beta_1-\beta_3+\gamma}s_{\alpha_2-\beta_2+\beta_1}s_{\beta_3-\gamma}s_{\beta_2}s_{\beta_1}s_{\alpha_3}s_{\beta_4}s_{\gamma} \xrightarrow{s_{\alpha_3}} (s_{\alpha_3}s_{\alpha_1+\beta_1-\beta_3+\gamma}s_{\alpha_2-\beta_2+\beta_1})(s_{\beta_3-\gamma}s_{\beta_2}s_{\beta_1}s_{\beta_4}s_{\gamma}).$$

see Fig. 2.9,(d). In the last step, s_{β_1} is replaced by $s_{-\beta_1}$, the corresponding diagram is depicted in Fig. 2.9,(e). Thus, the linkage diagram (5a) from Table 2.2 is equivalent to the Carter diagram $E_8(a_5)$.

2.3.6. Diagram $E_7(a_4)$. Case (6a). For the Carter diagram $E_7(a_4)$, there are 6 exceptional semi-Coxeter orbits with representatives (6a), (6b), (6c), (6d), (6e), (6f), that are similar to each other, see Table 2.2. Let us consider (6a).

Case (6a). First, reflections s_{β_3} and $s_{\alpha_1}s_{\alpha_3}$ are s -permuted. Two new connections $\{\beta_3 - \alpha_3 + \alpha_1, \beta_2\}$ and $\{\beta_3 - \alpha_3 + \alpha_1, \beta_4\}$ appear:

$$w = s_{\alpha_2}(s_{\alpha_1}s_{\alpha_3}s_{\beta_3})s_{\beta_1}s_{\beta_2}s_{\beta_4}s_{\gamma} = s_{\alpha_2}s_{\beta_3-\alpha_3+\alpha_1}s_{\alpha_1}s_{\alpha_3}s_{\beta_1}s_{\beta_2}s_{\beta_4}s_{\gamma} = s_{\beta_3-\alpha_3+\alpha_1}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\beta_1}s_{\beta_2}s_{\beta_4}s_{\gamma},$$

since α_2 and $\beta_3 - \alpha_3 + \alpha_1$ commute, see Fig. 2.10,(b). After that, reflections s_{β_2} and $s_{\alpha_2}s_{\alpha_3}$ are s -permuted. Then the connection $\{\beta_3 - \alpha_3 + \alpha_1, \beta_2\}$ disappears, and the connection $\{\beta_2 - \alpha_2 + \alpha_3, \beta_4\}$ appears:

$$w = s_{\beta_3-\alpha_3+\alpha_1}s_{\alpha_1}(s_{\alpha_2}s_{\alpha_3}s_{\beta_2})s_{\beta_1}s_{\beta_4}s_{\gamma} = s_{\beta_3-\alpha_3+\alpha_1}s_{\alpha_1}s_{\beta_2-\alpha_2+\alpha_3}s_{\alpha_2}s_{\alpha_3}s_{\beta_1}s_{\beta_4}s_{\gamma} = s_{\beta_3-\alpha_3+\alpha_1}s_{\beta_2-\alpha_2+\alpha_3}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\beta_1}s_{\beta_4}s_{\gamma},$$

since s_{α_1} and $s_{\beta_2-\alpha_2+\alpha_3}$ commute, see Fig. 2.10,(c). We have

$$(\beta_3 - \alpha_3 + \alpha_1, \beta_2 - \alpha_2 + \alpha_3) = (\beta_3, \alpha_3) - (\alpha_3, \alpha_3) - (\alpha_3, \beta_2) = \frac{1}{2} - 1 + \frac{1}{2} = 0.$$

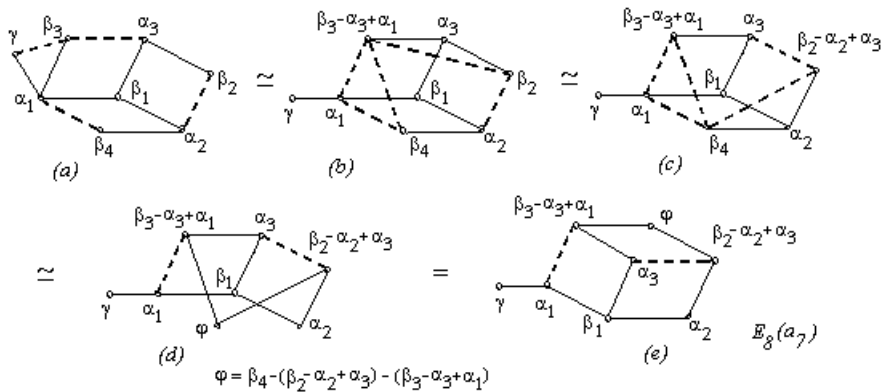


FIGURE 2.10. The linkage diagram (6a) for the Carter diagram $E_7(a_4)$

Further, reflections $s_{\beta_3-\alpha_3+\alpha_1}s_{\beta_2-\alpha_2+\alpha_3}$ and s_{β_4} are s -permuted:

$$\begin{aligned} w &= s_{\beta_3-\alpha_3+\alpha_1}s_{\beta_2-\alpha_2+\alpha_3}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\beta_1}s_{\beta_4}s_{\gamma} \stackrel{s_{\beta_4}}{\simeq} \\ & (s_{\beta_4}s_{\beta_3-\alpha_3+\alpha_1}s_{\beta_2-\alpha_2+\alpha_3})s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\beta_1}s_{\gamma} = s_{\beta_3-\alpha_3+\alpha_1}s_{\beta_2-\alpha_2+\alpha_3}s_{\varphi}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\beta_1}s_{\gamma}, \\ & \text{where } \varphi = \beta_4 - (\beta_3 - \alpha_3 + \alpha_1) - (\beta_2 - \alpha_2 + \alpha_3) = \beta_4 - \beta_3 - \beta_2 - \alpha_1 + \alpha_2. \end{aligned}$$

Then

$$\begin{aligned} (\varphi, \alpha_1) &= (\beta_4 - \beta_3 - \alpha_1, \alpha_1) = \frac{1}{2} + \frac{1}{2} - 1 = 0, \\ (\varphi, \alpha_2) &= (\beta_4 - \beta_2 + \alpha_2, \alpha_2) = -\frac{1}{2} - \frac{1}{2} + 1 = 0, \end{aligned}$$

i.e., connections $\{\varphi, \alpha_1\}$ and $\{\varphi, \alpha_2\}$ disappear, see Fig. 2.10,(d). Hence, the linkage diagram (6a) from Table 2.2 is equivalent to the Carter diagram $E_8(a_7)$, see Fig. 2.10,(e). \square

2.3.7. *Diagram D_7 . Cases (7a), (7b).* These 2 cases are different.

Case (7a). We have

$$w = s_{\gamma}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\beta_1}s_{\beta_2}s_{\beta_3} = s_{\alpha_3}(s_{\gamma+\alpha_3}s_{\alpha_1}s_{\alpha_2}s_{\alpha_4}s_{\beta_1})s_{\beta_2}s_{\beta_3}.$$

Let s -permute $s_{\gamma+\alpha_3}s_{\alpha_1}s_{\alpha_2}s_{\alpha_4}$ and s_{β_1} :

$$\begin{aligned} w &= s_{\alpha_3}s_{\beta_1+\alpha_1+\alpha_2+\gamma+\alpha_3}(s_{\gamma+\alpha_3}s_{\alpha_1}s_{\alpha_2}s_{\alpha_4})s_{\beta_2}s_{\beta_3} \stackrel{s_{\beta_2}s_{\beta_3}}{\simeq} \\ & (s_{\beta_2}s_{\beta_3}s_{\alpha_3}s_{\beta_1+\alpha_1+\alpha_2+\gamma+\alpha_3})(s_{\gamma+\alpha_3}s_{\alpha_1}s_{\alpha_2}s_{\alpha_4}). \end{aligned}$$

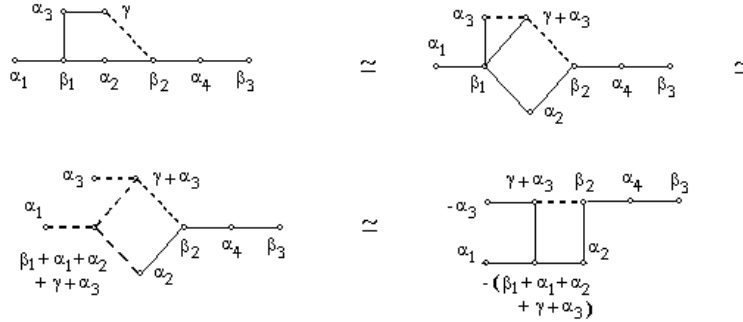


FIGURE 2.11. The linkage diagram (7a) for the Carter diagram D_7

Since

$$\begin{aligned} (\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3, \alpha_3) &= (\beta_1 + \gamma + \alpha_3, \alpha_3) = 1 - \frac{1}{2} - \frac{1}{2} = 0, \\ (\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3, \beta_2) &= (\gamma + \alpha_2, \beta_2) = \frac{1}{2} - \frac{1}{2} = 0, \end{aligned}$$

we get the bicolored decomposition, see Fig. 2.11. Since $s_{\beta_1+\alpha_1+\alpha_2+\gamma+\alpha_3} = s_{-(\beta_1+\alpha_1+\alpha_2+\gamma+\alpha_3)}$ and $s_{\alpha_3} = s_{-\alpha_3}$ we can change $\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3$ to the opposite vector $-(\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3)$ and α_3 to $-\alpha_3$. Thus, we obtain the last diagram in Fig. 2.11. The dotted edge $\{\gamma + \alpha_3, \beta_2\}$ (i.e., the property “be dotted edge”) can be moved to any edge of the square. Hence, we get the Carter diagram $E_8(a_2)$.

Case (7b). Let s -permute $s_{\gamma}s_{\alpha_2}$ and s_{β_1} as follows:

$$\begin{aligned}
 w = s_\gamma s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\beta_1} s_{\beta_2} s_{\beta_3} &\stackrel{s_\gamma s_{\alpha_2}}{\simeq} (s_{\alpha_1} s_{\alpha_3} s_{\alpha_4}) s_{\beta_2} s_{\beta_3} (s_{\beta_1} s_\gamma s_{\alpha_2}) = \\
 s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\beta_2} s_{\beta_3} s_\gamma s_{\alpha_2} s_{\beta_1 - \gamma + \alpha_2} &\stackrel{s_{\beta_1 - \gamma + \alpha_2}}{\simeq} s_{\alpha_4} (s_{\beta_1 - \gamma + \alpha_2} s_{\alpha_1} s_{\beta_2} s_{\beta_3}) (s_{\alpha_3} s_\gamma s_{\alpha_2}) \stackrel{s_{\alpha_4}}{\simeq} \\
 (s_{\beta_1 - \gamma + \alpha_2} s_{\alpha_1} s_{\beta_2} s_{\beta_3}) (s_{\alpha_3} s_\gamma s_{\alpha_2} s_{\alpha_4}), &
 \end{aligned}$$

where

$$\begin{aligned}
 (\beta_1 - \gamma + \alpha_2, \alpha_1) &= (\beta_1 - \gamma, \alpha_1) = -\frac{1}{2} + \frac{1}{2} = 0, \\
 (\beta_1 - \gamma + \alpha_2, \beta_2) &= (-\gamma + \alpha_2, \beta_2) = \frac{1}{2} - \frac{1}{2} = 0.
 \end{aligned}$$



FIGURE 2.12. The linkage diagram (7a) for the Carter diagram D_7

Thus, as in the Case (7a), w has the bicolored decomposition corresponding to the Carter diagram $E_8(a_2)$, see Fig. 2.12. □

2.4. Finding semi-Coxeter orbits.

2.4.1. *How to find semi-Coxeter orbits?* We use two ways to find semi-Coxeter orbits. The first one is the matrix approach:

- 1) Calculation of powers of dual semi-Coxeter elements \mathbf{c}^* , see Tables A.3 - A.5.
- 2) Applying $(\mathbf{c}^*)^k$ to any unicolored linkage diagram γ^\vee until finding the period of \mathbf{c}^* on this linkage.
- 3) Search any new linkage from the corresponding linkage system, preferably unicolored and back to step 2). The linkage systems for all Carter diagrams from DE4 and C4 are in [St10.II].

The second way is the diagram approach. We find all semi-Coxeter orbits as a closed cycles in the linkage system. We call such a semi-Coxeter orbit the \mathbf{c}^* -cycle. The link connecting γ^\vee and $\mathbf{c}\gamma^\vee$ we call the \mathbf{c}^* -transition. Every \mathbf{c}^* -cycle consists of \mathbf{c}^* -transitions $\gamma^\vee \rightarrow \mathbf{c}\gamma^\vee$. Each \mathbf{c}^* -transition consists of 2 passages, one after the other: $\gamma^\vee \rightarrow {}^t w_\beta \gamma^\vee$ and ${}^t w_\beta \gamma^\vee \rightarrow {}^t w_\alpha {}^t w_\beta \gamma^\vee$, where

$${}^t w_\alpha = \prod_{i=1}^k s_{\alpha_i}^*, \quad {}^t w_\beta = \prod_{j=1}^h s_{\beta_j}^*,$$

Reflections $s_{\beta_j}^*$ and $s_{\alpha_i}^*$ act on the linkage diagrams in the linkage system $\mathcal{L}(\Gamma)$. The order of actions of $s_{\beta_j}^*$ within ${}^t w_\beta$ (resp. $s_{\alpha_i}^*$ within ${}^t w_\alpha$) does not matter since all $s_{\beta_j}^*$ (resp. $s_{\alpha_i}^*$) mutually commute. Examples of semi-Coxeter orbits are presented in Appendix B, where the orbits are differed by colors or bold and dotted lines. Let $\mathcal{L}(\Gamma)$ be the linkage system for the Carter diagram of Γ . Note that for any linkage $\gamma^\vee \in \mathcal{L}(\Gamma)$, we have $-\gamma^\vee \in \mathcal{L}(\Gamma)$, since $\mathcal{B}_L^\vee(\gamma^\vee) = \mathcal{B}_L^\vee(-\gamma^\vee)$ and

$$\mathcal{B}_L^\vee(\gamma^\vee) < 2 \iff \gamma^\vee \in \mathcal{L}(\Gamma),$$

see [St10.II, Theorem 2.14]. Two orbits are said to be the *opposite orbits* if for every linkage γ^\vee in one of the orbits there exists the linkage $-\gamma^\vee$ in another one. There are some orbits which are opposite to themselves, such an orbit is said to be the *self-opposite orbit*.

For Carter diagrams D_4 , $D_4(a_1)$, $D_5(a_1)$, D_5 , $E_6(a_1)$, $E_6(a_2)$, E_6 , the figures of linkage systems with semi-Coxeter orbits are depicted in Fig. B.13-B.19.

APPENDIX A. The dual semi-Coxeter element for the Carter diagrams







The Carter diagram	The transpose semi-Coxeter element ${}^t\mathbf{c}$	The dual semi-Coxeter element $\mathbf{c}^* = {}^t\mathbf{c}^{-1}$	Order of ${}^t\mathbf{c}$
 $\mathbf{D}_4(\mathbf{a}_1)$	$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$	4
 $\mathbf{D}_5(\mathbf{a}_1) = \mathbf{D}_5(\mathbf{a}_2)$	$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix}$	12
 $\mathbf{E}_6(\mathbf{a}_1)$	$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$	9
 $\mathbf{E}_6(\mathbf{a}_2)$	$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$	6
 $\mathbf{D}_6(\mathbf{a}_1) = \mathbf{D}_6(\mathbf{a}_3)$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	8
 $\mathbf{D}_6(\mathbf{a}_2)$	$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 1 & 0 & 2 \end{bmatrix}$	6

TABLE A.3. The dual semi-Coxeter element \mathbf{c}^* for $l < 7$

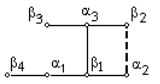
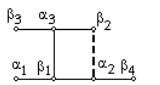

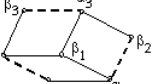
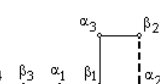
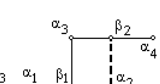
The Carter diagram	The transpose semi-Coxeter element ${}^t\mathbf{c}$	The dual semi-Coxeter element $\mathbf{c}^* = {}^t\mathbf{c}^{-1}$	Order of ${}^t\mathbf{c}$
 $E_7(a_1)$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$	14
 $E_7(a_2)$	$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$	12
 $E_7(a_3)$	$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$	30
 $E_7(a_4)$	$\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$	6
 $D_7(a_1) = D_7(a_4)$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$	20
 $D_7(a_2) = D_7(a_3)$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	24

TABLE A.4. (cont.) The dual semi-Coxeter element \mathbf{c}^* for $l = 7$


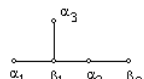
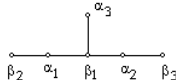
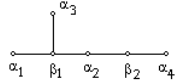

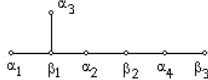
The Carter diagram	The transpose semi-Coxeter element ${}^t\mathbf{c}$	The dual semi-Coxeter element $\mathbf{c}^* = {}^t\mathbf{c}^{-1}$	Order of ${}^t\mathbf{c}$
 D_4	$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$	6
 D_5	$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$	8
 E_6	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$	12
 D_6	$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$	10
 E_7	$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$	18
 D_7	$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$	12

TABLE A.5. (cont.) The dual semi-Coxeter element \mathbf{c}^*

APPENDIX B. Semi-Coxeter orbits

Recall, that orbits of dual semi-Coxeter element acting on the linkage diagrams are said to be *semi-Coxeter orbits*, see Section 2.2.2.

B.1. Semi-Coxeter orbits for $D_l(a_i)$, $E_l(a_i)$, D_l , E_l , where $l < 7$.


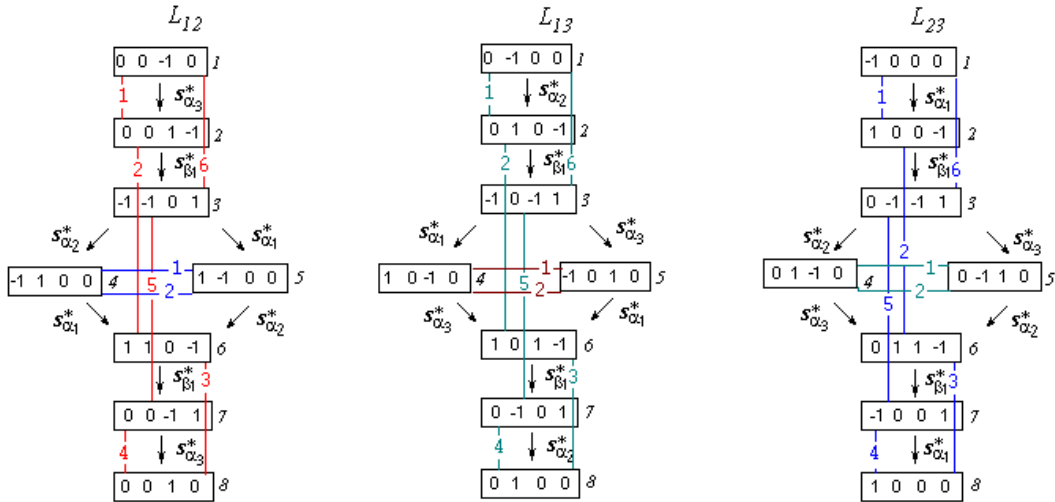
 D_4	Orbit 1 (red ¹ , L_{12})	Orbit 2 (green, L_{13})	Orbit 3 (blue, L_{23})
γ^\vee	$[0, 0, -1, 0]$	$[0, -1, 0, 0]$	$[-1, 0, 0, 0]$
$c^*\gamma^\vee$	$[0, 0, 1, -1]$	$[0, 1, 0, -1]$	$[1, 0, 0, -1]$
$(c^*)^2\gamma^\vee$	$[1, 1, 0, -1]$	$[1, 0, 1, -1]$	$[0, 1, 1, -1]$
$(c^*)^3\gamma^\vee$	$[0, 0, 1, 0]$	$[0, 1, 0, 0]$	$[1, 0, 0, 0]$
$(c^*)^4\gamma^\vee$	$[0, 0, -1, 1]$	$[0, -1, 0, 1]$	$[-1, 0, 0, 1]$
$(c^*)^5\gamma^\vee$	$[-1, -1, 0, 1]$	$[-1, 0, -1, 1]$	$[0, -1, -1, 1]$
	Orbit 4 (blue, L_{12})	Orbit 5 (brown, L_{13})	Orbit 6 (green, L_{23})
γ^\vee	$[-1, 1, 0, 0]$	$[1, 0, -1, 0]$	$[0, 1, -1, 0]$
$(c^*)\gamma^\vee$	$[1, -1, 0, 0]$	$[-1, 0, 1, 0]$	$[0, -1, 1, 0]$

TABLE B.6. D_4 , there exist 6 semi-Coxeter orbits. All orbits are self-oppositeFIGURE B.13. D_4 , 6 semi-Coxeter orbits, three of length 6, three of length 2

¹Here and below in all tables unicolored linkage labels vectors are framed by a rectangle.

$\begin{array}{c} \beta_2 \quad \alpha_2 \\ \alpha_1 \quad \beta_1 \end{array} \quad D_4(a_1)$	Orbit 1 (green, II)	Orbit 2 (red, II)	Orbit 3 (brown, III)
γ^\vee	$\boxed{1, 0, 0, 0}$	$0, 1, 1, -1$	$\boxed{0, 0, 0, 1}$
$c^* \gamma^\vee$	$1, 0, -1, -1$	$\boxed{0, 1, 0, 0}$	$-1, -1, 0, 1$
$(c^*)^2 \gamma^\vee$	$\boxed{-1, 0, 0, 0}$	$0, -1, -1, 1$	$\boxed{0, 0, 0, -1}$
$(c^*)^3 \gamma^\vee$	$-1, 0, 1, 1$	$\boxed{0, -1, 0, 0}$	$1, 1, 0, -1$
	Orbit 4 (green, III)	Orbit 5 (red, I) (no unicolored diagrams)	Orbit 6 (blue, I) (no unicolored diagrams)
γ^\vee	$1, -1, -1, 0$	$0, 1, 1, 0$	$1, 0, -1, 0$
$(c^*) \gamma^\vee$	$\boxed{0, 0, 1, 0}$	$-1, 0, 0, 1$	$0, -1, 0, 1$
$(c^*)^2 \gamma^\vee$	$-1, 1, 1, 0$	$0, -1, -1, 0$	$-1, 0, 1, 0$
$(c^*)^3 \gamma^\vee$	$\boxed{0, 0, -1, 0}$	$1, 0, 0, -1$	$0, 1, 0, -1$

TABLE B.7. $D_4(a_1)$, there exist 6 semi-Coxeter orbits. All orbits are self-opposite. Orbits 1 – 4 contain unicolored linkage diagrams.

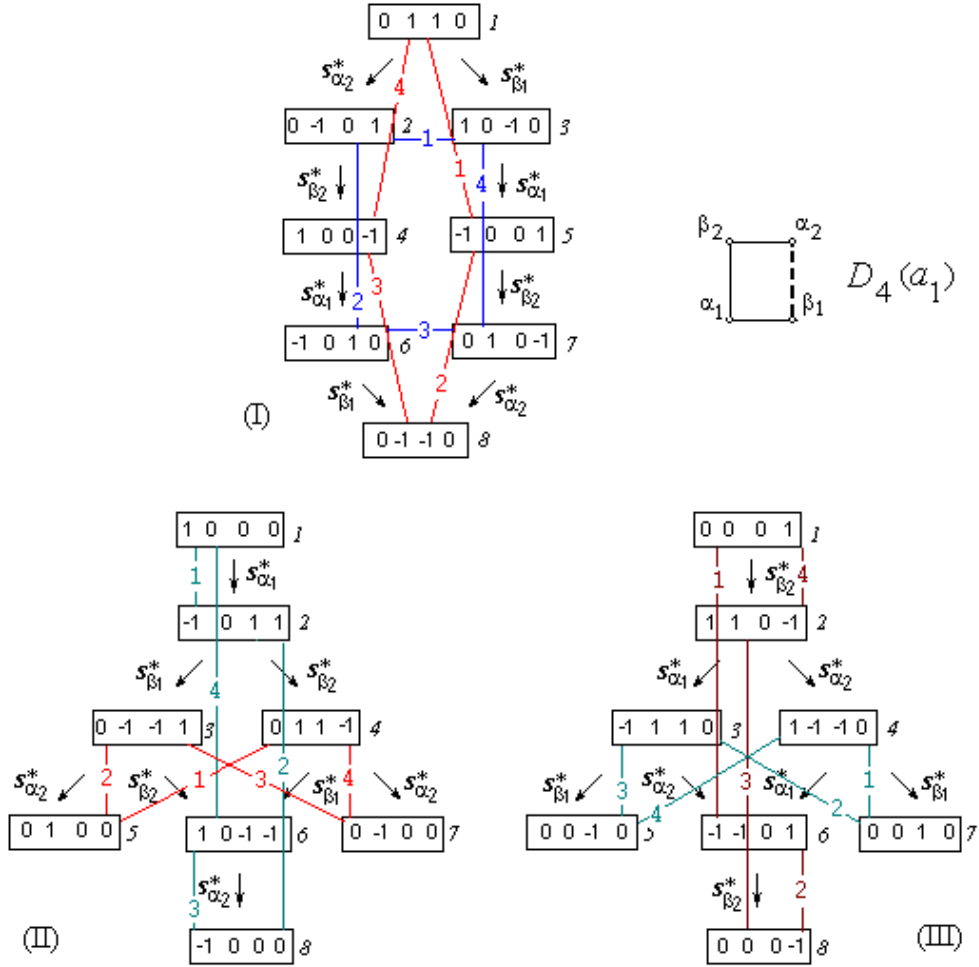


FIGURE B.14. The linkage system of $D_4(a_1)$, three components, 24 linkage diagrams, 3 components

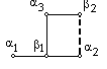
 $D_5(a_1) = D_5(a_2)$	Orbit 1 (red)	Orbit 2 (green)	Orbit 3 (brown) (self-opposite)	Orbit 4 (blue) (self-opposite)
γ^\vee $(c^*)\gamma^\vee$ $(c^*)^2\gamma^\vee$ $(c^*)^3\gamma^\vee$ $(c^*)^4\gamma^\vee$ $(c^*)^5\gamma^\vee$ $(c^*)^6\gamma^\vee$ $(c^*)^7\gamma^\vee$ $(c^*)^8\gamma^\vee$ $(c^*)^9\gamma^\vee$ $(c^*)^{10}\gamma^\vee$ $(c^*)^{11}\gamma^\vee$	$[0, 0, -1, 0, 0]$ $0, 0, 1, -1, -1$ $1, 0, 1, -1, 0$ $0, 1, 0, 0, 1$ $0, 0, -1, 1, 0$ $-1, -1, 0, 1, -1$ $[0, -1, 0, 0, 0]$ $0, 1, 0, -1, 1$ $1, 1, 0, -1, 0$ $0, 0, 1, 0, -1$ $0, -1, 0, 1, 0$ $-1, 0, -1, 1, 1$	$[-1, 1, 0, 0, 0]$ $1, -1, 0, 0, -1$ $[-1, 0, 1, 0, 0]$ $1, 0, -1, 0, 1$	$[-1, 0, 0, 0, 0]$ $1, 0, 0, -1, 0$ $0, 1, 1, -1, 0$ $[1, 0, 0, 0, 0]$ $-1, 0, 0, 1, 0$ $0, -1, -1, 1, 0$	$[0, 0, 0, 0, -1]$ $0, -1, 1, 0, -1$ $[0, 0, 0, 0, 1]$ $0, 1, -1, 0, 1$

TABLE B.8. $D_5(a_1) = D_5(a_2)$, 6 semi-Coxeter orbits. All orbits contain unicolored linkage diagrams. Orbits 1, 2 have opposite orbits (in bottom component). Orbits 3, 4 are self-opposite

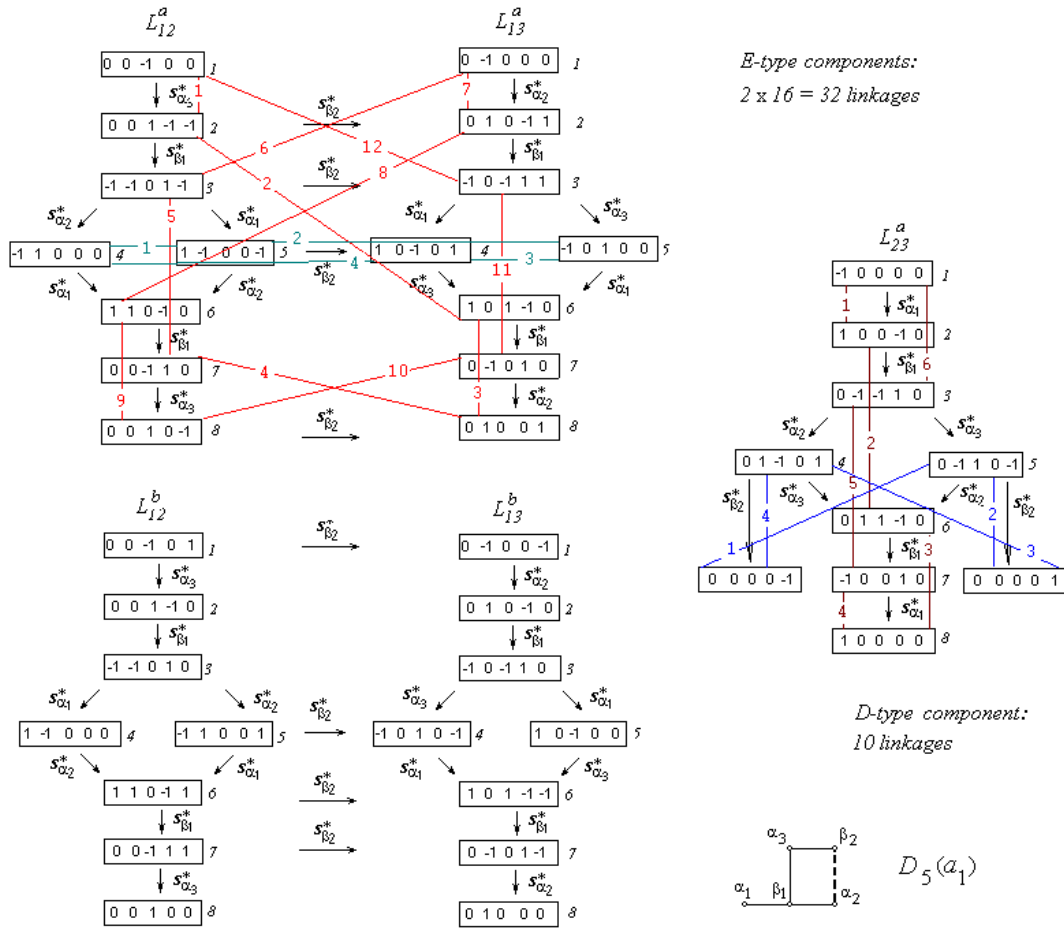


FIGURE B.15. The linkage system of $D_5(a_1)$, 42 linkage diagrams, 3 components

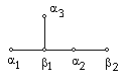
 D_5	Orbit 1 (red, E -type)	Orbit 2 (blue, E -type)	Orbit 3 (red, D -type) (self-opposite)	Orbit 4 (blue, D -type) (self-opposite)
γ^\vee $(c^*)\gamma^\vee$ $(c^*)^2\gamma^\vee$ $(c^*)^3\gamma^\vee$ $(c^*)^4\gamma^\vee$ $(c^*)^5\gamma^\vee$ $(c^*)^6\gamma^\vee$ $(c^*)^7\gamma^\vee$	$[0, 0, -1, 0, 0]$ $0, 0, 1, -1, 0$ $1, 1, 0, -1, -1$ $0, 1, 1, -1, 0$ $[1, 0, 0, 0, 0]$ $-1, 0, 0, 1, 0$ $0, -1, -1, 1, 1$ $-1, -1, 0, 1, 0$	$0, 0, 1, 0, -1$ $[0, 1, -1, 0, 0]$ $0, -1, 1, 0, 1$ $0, 0, -1, 1, -1$ $-1, 0, 0, 0, 1$ $[1, -1, 0, 0, 0]$ $-1, 1, 0, 0, -1$ $1, 0, 0, -1, 1$	$[0, 0, 0, 0, -1]$ $0, 1, 0, -1, 0$ $1, 0, 1, -1, 0$ $0, 1, 0, 0, -1$ $[0, 0, 0, 0, 1]$ $0, -1, 0, 1, 0$ $-1, 0, -1, 1, 0$ $0, -1, 0, 0, 1$	$[1, 0, -1, 0, 0]$ $[-1, 0, 1, 0, 0]$

TABLE B.9. D_5 , there exist 6 semi-Coxeter orbits. All orbits contain unicolored linkage diagrams. Orbits 1, 2 have opposite orbits (in 2nd E -type component). Orbits 3, 4 are self-opposite

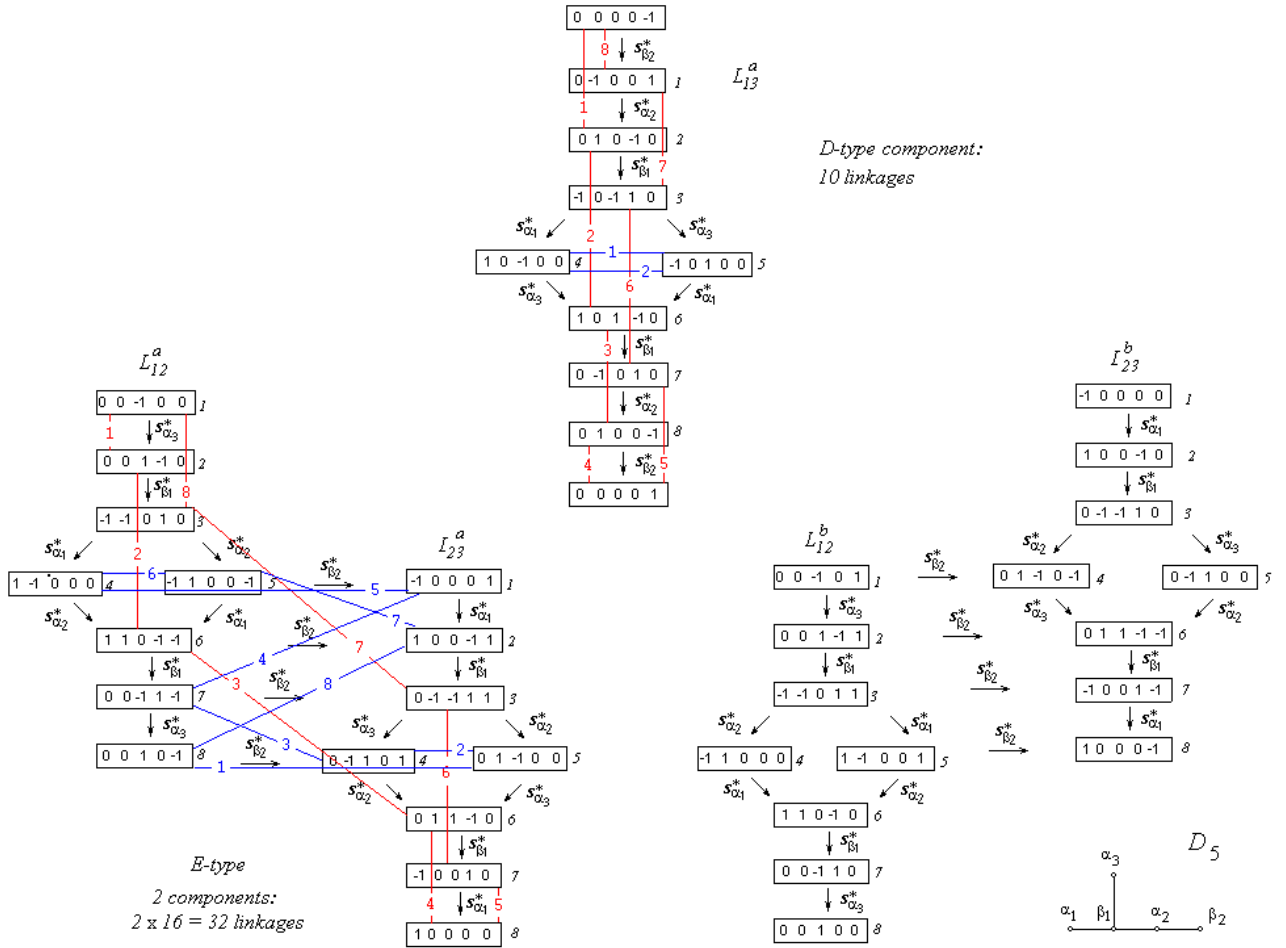


FIGURE B.16. The linkage system of D_5 , 42 linkage diagrams, 3 components

$\begin{array}{c} \beta_3 \quad \alpha_3 \quad \beta_2 \\ \alpha_1 \quad \beta_1 \quad \alpha_2 \end{array}$ $\mathbf{E}_6(\mathbf{a}_1)$	Orbit 1 (red)	Orbit 2 (green)	Orbit 2 (blue)
γ^\vee	$[0, 0, 0, 0, 0, -1]$	$[0, 0, 0, 0, 1, 0]$	$[0, 0, 0, 0, -1, 1]$
$\mathbf{c}^* \gamma^\vee$	$[0, 0, 1, -1, -1, 0]$	$[0, 1, -1, 0, 1, 1]$	$[0, -1, 0, 1, 0, -1]$
$(\mathbf{c}^*)^2 \gamma^\vee$	$[1, 0, 1, -1, 0, -1]$	$[0, 0, -1, 1, 0, 0]$	$[-1, 0, 0, 0, 0, 1]$
$(\mathbf{c}^*)^3 \gamma^\vee$	$[0, 1, 1, -1, 0, 0]$	$[-1, -1, 0, 1, -1, 0]$	$[1, 0, -1, 0, 1, 0]$
$(\mathbf{c}^*)^4 \gamma^\vee$	$[1, 0, 0, 0, 0, 0]$	$[0, -1, 0, 0, 0, 0]$	$[-1, 1, 0, 0, 0, 0]$
$(\mathbf{c}^*)^5 \gamma^\vee$	$[-1, 0, 0, 1, 0, 0]$	$[0, 1, 0, -1, 1, 0]$	$[1, -1, 0, 0, -1, 0]$
$(\mathbf{c}^*)^6 \gamma^\vee$	$[0, -1, -1, 1, 0, 1]$	$[1, 1, 0, -1, 0, 0]$	$[-1, 0, 1, 0, 0, -1]$
$(\mathbf{c}^*)^7 \gamma^\vee$	$[-1, 0, -1, 1, 1, 0]$	$[0, 0, 1, 0, -1, -1]$	$[1, 0, 0, -1, 0, 1]$
$(\mathbf{c}^*)^8 \gamma^\vee$	$[0, 0, -1, 0, 0, 1]$	$[0, -1, 1, 0, -1, 0]$	$[0, 1, 0, 0, 1, -1]$

TABLE B.10. $\mathbf{E}_6(\mathbf{a}_1)$, there exist 6 semi-Coxeter orbits, each of length 9. Every orbit contains the β -unicolored linkage diagram γ^\vee . Orbits 1, 2, 3 have opposite orbits (starting from $-\gamma^\vee$)

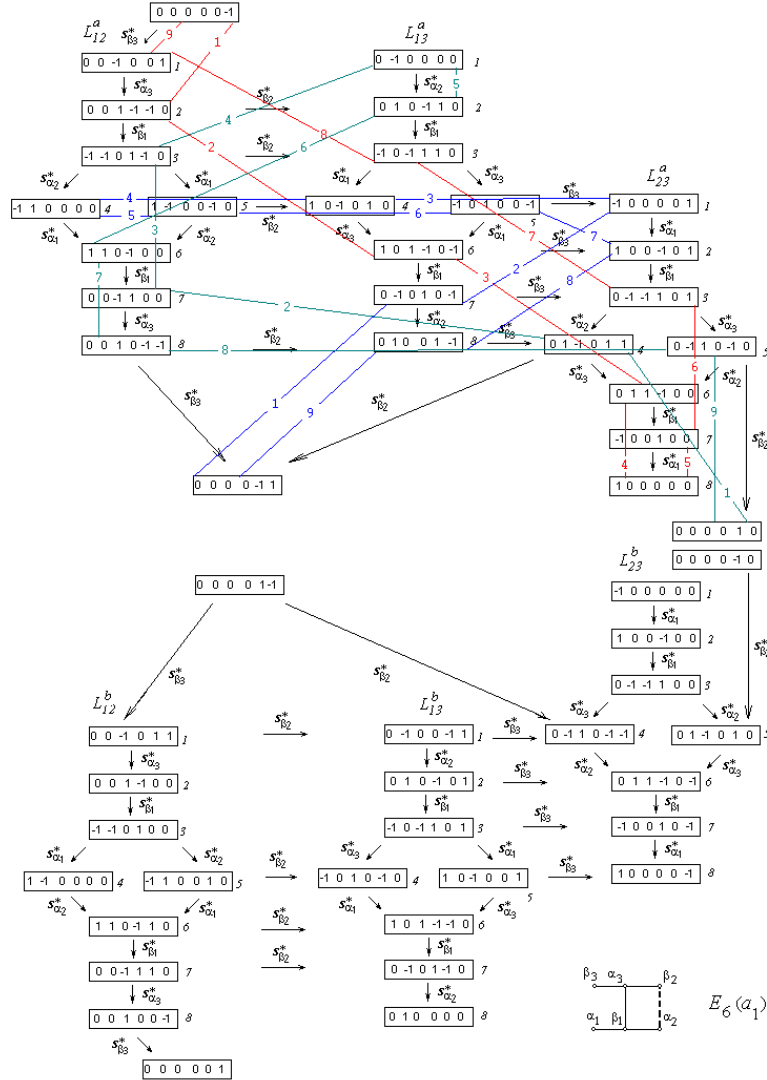


FIGURE B.17. The linkage system of $E_6(a_1)$, two components, 54 linkage diagrams, 6 loctets

	$\begin{array}{c} \beta_3 \quad \alpha_3 \quad \beta_2 \\ \alpha_1 \quad \beta_1 \quad \alpha_2 \end{array} \quad E_6(a_2)$	Orbit 1 (blue)	Orbit 2 (green)
γ^\vee		$[0, -1, 0, 0, 0, 0]$	$[0, 0, 0, 0, -1, 0]$
$c^* \gamma^\vee$		$[0, 1, 0, -1, 1, 0]$	$[0, -1, 1, 0, -1, -1]$
$(c^*)^2 \gamma^\vee$		$[1, 1, 0, -1, 0, 1]$	$[-1, 0, 1, 0, 0, -1]$
$(c^*)^3 \gamma^\vee$		$[1, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 1]$
$(c^*)^4 \gamma^\vee$		$[-1, 0, 0, 1, 0, -1]$	$[1, 0, -1, 0, 1, 1]$
$(c^*)^5 \gamma^\vee$		$[-1, -1, 0, 1, -1, 0]$	$[0, 1, -1, 0, 1, 0]$
	Orbit 3 (red) (no unicolored diagrams)	Orbit 4 (brown) (no unicolored diagrams)	Orbit 5 (turquoise)
γ^\vee	$[-1, 0, 0, 0, 0, -1]$	$[0, 0, -1, 0, 0, 1]$	$[0, 0, 0, 0, 1, -1]$
$c^* \gamma^\vee$	$[0, 0, 1, -1, -1, 0]$	$[1, 0, 0, -1, 0, 0]$	$[-1, 1, 0, 0, 0, 0]$
$(c^*)^2 \gamma^\vee$	$[1, 0, 1, -1, 0, 0]$	$[0, 1, 1, -1, 0, -1]$	$[1, -1, 0, 0, -1, 1]$
$(c^*)^3 \gamma^\vee$	$[0, 1, 0, 0, 1, 0]$	$[0, 0, 1, 0, -1, 0]$	
$(c^*)^4 \gamma^\vee$	$[0, 0, -1, 1, 0, 1]$	$[0, -1, 0, 1, 0, 0]$	
$(c^*)^5 \gamma^\vee$	$[0, -1, -1, 1, 0, 0]$	$[-1, 0, -1, 1, 1, 0]$	

TABLE B.11. $E_6(a_2)$, there exist 10 semi-Coxeter orbits. Orbits 1 – 5 have opposite orbits. Only orbits 1, 2, 5 contain unicolored linkage diagrams

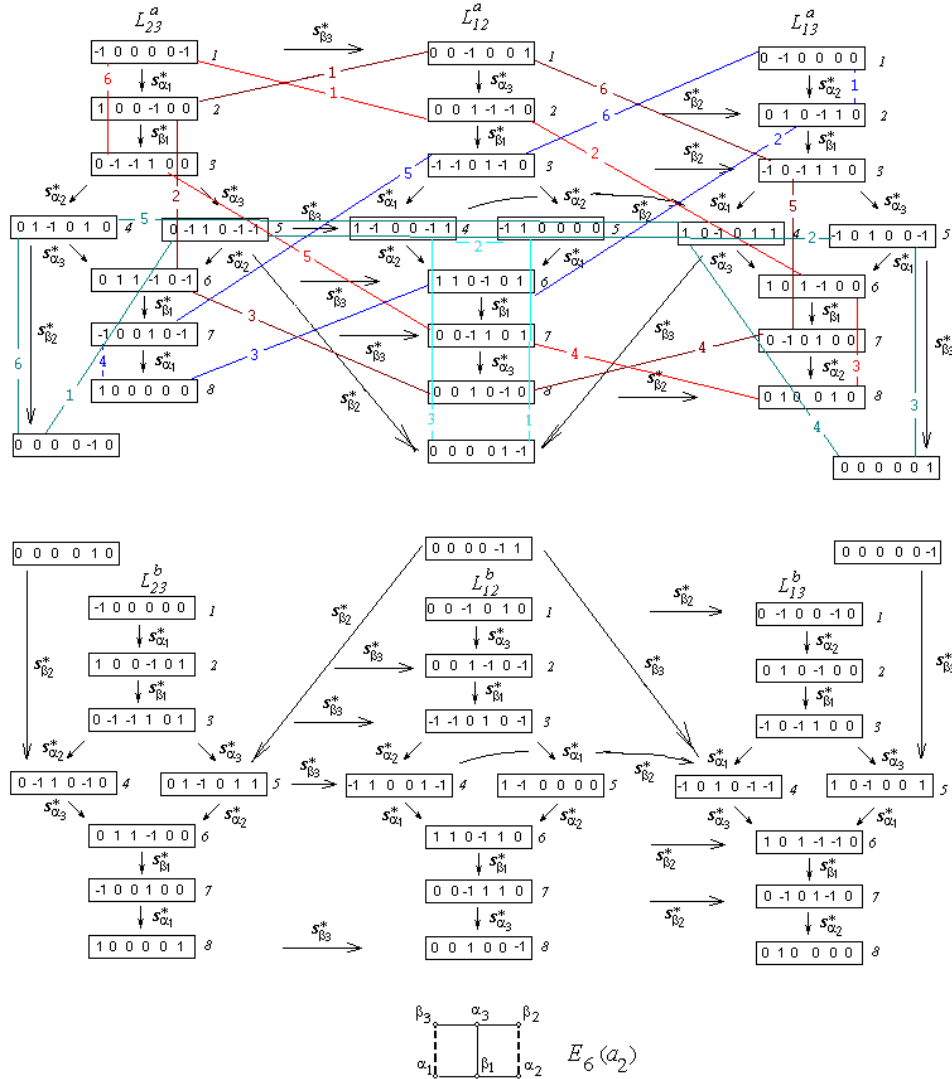


FIGURE B.18. The linkage system of $E_6(a_2)$, two components, 54 linkage diagrams, 6 loctets

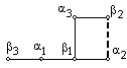
 $\mathbf{D_6(a_1)}$	Orbit 1 (no unicolored)	Orbit 2 (no unicolored)	Orbit 3 (no unicolored)	Orbit 4 (no unicolored)
γ^\vee	0, -1, 0, 0, -1, 0	0, 0, -1, 0, 1, 0	1, 0, -1, 0, 0, -1	1, -1, 0, 0, 0, -1
$\mathbf{c^*}\gamma^\vee$	0, 0, 1, -1, 0, 0	0, 1, 0, -1, 0, 0	0, 0, 1, -1, -1, 1	0, 1, 0, -1, 1, 1
$(\mathbf{c^*})^2\gamma^\vee$	1, 1, 0, -1, 1, -1	1, 0, 1, -1, -1, -1	0, 0, 1, 0, 0, -1	0, 1, 0, 0, 0, -1
$(\mathbf{c^*})^3\gamma^\vee$	1, 1, 0, -1, 0, 0	1, 0, 1, -1, 0, 0	1, 0, -1, 0, 1, 0	1, -1, 0, 0, -1, 0
$(\mathbf{c^*})^4\gamma^\vee$	0, 0, 1, 0, -1, 0	0, 1, 0, 0, 1, 0	-1, 1, 0, 0, 0, 1	-1, 0, 1, 0, 0, 1
$(\mathbf{c^*})^5\gamma^\vee$	0, -1, 0, 1, 0, 0	0, 0, -1, 1, 0, 0	0, -1, 0, 1, -1, -1	0, 0, -1, 1, 1, -1
$(\mathbf{c^*})^6\gamma^\vee$	-1, 0, -1, 1, 1, 1	-1, -1, 0, 1, -1, 1	0, -1, 0, 0, 0, 1	0, 0, -1, 0, 0, 1
$(\mathbf{c^*})^7\gamma^\vee$	-1, 0, -1, 1, 0, 0	-1, -1, 0, 1, 0, 0	-1, 1, 0, 0, 1, 0	-1, 0, 1, 0, -1, 0
	Orbit 5	Orbit 6	Orbit 7	Orbit 8
γ^\vee	0, 0, 0, 0, 0, -1	0, 0, -1, 0, 0, 0	0, -1, 0, 0, 0, 0	-1, 1, 0, 0, 0, 0
$\mathbf{c^*}\gamma^\vee$	1, 0, 0, -1, 0, 0	0, 0, 1, -1, -1, 0	0, 1, 0, -1, 1, 0	1, -1, 0, 0, -1, -1
$(\mathbf{c^*})^2\gamma^\vee$	0, 1, 1, -1, 0, 0	1, 0, 1, -1, 0, -1	1, 1, 0, -1, 0, -1	0, 0, 1, -1, 0, 1
$(\mathbf{c^*})^3\gamma^\vee$	1, 0, 0, 0, 0, -1	1, 1, 0, -1, 1, 0	1, 0, 1, -1, -1, 0	0, 1, 0, 0, 1, -1
$(\mathbf{c^*})^4\gamma^\vee$	0, 0, 0, 0, 0, 1	0, 1, 0, 0, 0, 0	0, 0, 1, 0, 0, 0	1, 0, -1, 0, 0, 0
$(\mathbf{c^*})^5\gamma^\vee$	-1, 0, 0, 1, 0, 0	0, -1, 0, 1, -1, 0	0, 0, -1, 1, 1, 0	-1, 0, 1, 0, -1, 1
$(\mathbf{c^*})^6\gamma^\vee$	0, -1, -1, 1, 0, 0	-1, -1, 0, 1, 0, 1	-1, 0, -1, 1, 0, 1	0, -1, 0, 1, 0, -1
$(\mathbf{c^*})^7\gamma^\vee$	-1, 0, 0, 0, 0, 1	-1, 0, -1, 1, 1, 0	-1, -1, 0, 1, -1, 0	0, 0, -1, 0, 1, 1
		Orbit 9 (brown, right)	Orbit 10 (blue, dotted)	
γ^\vee		-1, 0, 1, 0, 0, 0	0, 1, -1, 0, 1, 0	
$\mathbf{c^*}\gamma^\vee$		1, 0, -1, 0, 1, -1	0, 0, 0, 0, -1, 0	
$(\mathbf{c^*})^2\gamma^\vee$		0, 1, 0, -1, 0, 1	0, -1, 1, 0, -1, 0	
$(\mathbf{c^*})^3\gamma^\vee$		0, 0, 1, 0, -1, -1	0, 0, 0, 0, 1, 0	
$(\mathbf{c^*})^4\gamma^\vee$		1, -1, 0, 0, 0, 0		
$(\mathbf{c^*})^5\gamma^\vee$		-1, 1, 0, 0, 1, 1		
$(\mathbf{c^*})^6\gamma^\vee$		0, 0, -1, 1, 0, -1		
$(\mathbf{c^*})^7\gamma^\vee$		0, -1, 0, 0, -1, 1		

TABLE B.13. $\mathbf{D_6(a_1)}$, there exist 10 semi-Coxeter orbits: nine of length 8, and one of length 4. Pairs of orbits $\{1, 2\}$, $\{3, 4\}$, $\{6, 7\}$, $\{8, 9\}$ are pairs of opposite orbits. Orbits 5 and 10 are self-opposite

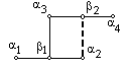
 $\mathbf{D}_6(\mathbf{a}_2)$	Orbit 1	Orbit 2 (no unicolored)	Orbit 3	Orbit 4 (no unicolored)	Orbit 5
γ^\vee	$[0, 0, -1, 0, 0, 0]$	$0, 0, -1, 0, 0, 1$	$[0, -1, 0, -1, 0, 0]$	$0, -1, 0, 1, 0, -1$	$1, -1, 0, 0, 0, -1$
$\mathbf{c}^*\gamma^\vee$	$0, 0, 1, 0, -1, -1$	$0, 1, 0, -1, -1, 1$	$0, 1, 0, 1, -1, 0$	$0, 0, 1, 0, -1, 0$	$-1, 0, 1, 1, 0, -1$
$(\mathbf{c}^*)^2\gamma^\vee$	$1, 0, 1, 1, -1, -1$	$1, 1, 0, 0, -1, 0$	$1, 0, 1, -1, -1, 0$	$1, 1, 0, 0, -1, 1$	$[1, -1, 0, 0, 0, 0]$
$(\mathbf{c}^*)^3\gamma^\vee$	$[0, 0, 1, 0, 0, 0]$	$0, 0, 1, 0, 0, -1$	$[0, 1, 0, 1, 0, 0]$	$0, 1, 0, -1, 0, 1$	$-1, 1, 0, 0, 0, 1$
$(\mathbf{c}^*)^4\gamma^\vee$	$0, 0, -1, 0, 1, 1$	$0, -1, 0, 1, 1, -1$	$0, -1, 0, -1, 1, 0$	$0, 0, -1, 0, 1, 0$	$1, 0, -1, -1, 0, 1$
$(\mathbf{c}^*)^5\gamma^\vee$	$-1, 0, -1, -1, 1, 1$	$-1, -1, 0, 0, 1, 0$	$-1, 0, -1, 1, 1, 0$	$-1, -1, 0, 0, 1, -1$	$[-1, 1, 0, 0, 0, 0]$
	Orbit 6	Orbit 7 (no unicolored)	Orbit 8	Orbit 9 (no unicolored)	Orbit 10
γ^\vee	$[0, -1, 0, 0, 0, 0]$	$0, -1, 0, 0, 0, -1$	$[0, 0, -1, 1, 0, 0]$	$0, 0, -1, -1, 0, 1$	$1, 0, -1, 0, 0, 1$
$\mathbf{c}^*\gamma^\vee$	$0, 1, 0, 0, -1, 1$	$0, 0, 1, 1, -1, -1$	$0, 0, 1, -1, -1, 0$	$0, 1, 0, 0, -1, 0$	$-1, 1, 0, -1, 0, 1$
$(\mathbf{c}^*)^2\gamma^\vee$	$1, 1, 0, -1, -1, 1$	$1, 0, 1, 0, -1, 0$	$1, 1, 0, 1, -1, 0$	$1, 0, 1, 0, -1, -1$	$[1, 0, -1, 0, 0, 0]$
$(\mathbf{c}^*)^3\gamma^\vee$	$[0, 1, 0, 0, 0, 0]$	$0, 1, 0, 0, 0, 1$	$[0, 0, 1, -1, 0, 0]$	$0, 0, 1, 1, 0, -1$	$-1, 0, 1, 0, 0, -1$
$(\mathbf{c}^*)^4\gamma^\vee$	$0, -1, 0, 0, 1, -1$	$0, 0, -1, -1, 1, 1$	$0, 0, -1, 1, 1, 0$	$0, -1, 0, 0, 1, 0$	$1, -1, 0, 1, 0, -1$
$(\mathbf{c}^*)^5\gamma^\vee$	$-1, -1, 0, 1, 1, -1$	$-1, 0, -1, 0, 1, 0$	$-1, -1, 0, -1, 1, 0$	$-1, 0, -1, 0, 1, 1$	$[-1, 0, 1, 0, 0, 0]$
	Orbit 11	Orbit 12	Orbit 13	Orbit 14	
γ^\vee	$[-1, 0, 0, 0, 0, 0]$	$0, 1, -1, 0, 0, 1$	$[-1, 0, 1, -1, 0, 0]$	$[-1, 1, 0, 1, 0, 0]$	
$\mathbf{c}^*\gamma^\vee$	$1, 0, 0, 0, -1, 0$	$[0, 0, 0, -1, 0, 0]$	$[1, 0, -1, 1, 0, 0]$	$[1, -1, 0, -1, 0, 0]$	
$(\mathbf{c}^*)^2\gamma^\vee$	$0, 1, 1, 0, -1, 0$	$0, 0, 0, 1, 0, -1$			
$(\mathbf{c}^*)^3\gamma^\vee$	$[1, 0, 0, 0, 0, 0]$	$0, -1, 1, 0, 0, -1$			
$(\mathbf{c}^*)^4\gamma^\vee$	$-1, 0, 0, 0, 1, 0$	$[0, 0, 0, 1, 0, 0]$			
$(\mathbf{c}^*)^5\gamma^\vee$	$0, -1, -1, 0, 1, 0$	$0, 0, 0, -1, 0, 1$			

TABLE B.14. $\mathbf{D}_6(\mathbf{a}_2)$, there exist 14 semi-Coxeter orbits: orbits 1 – 12 are of length 6, and orbits 13, 14 are of length 2. All orbits are self-opposite

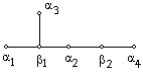
 \mathbf{D}_6	Orbit 1	Orbit 2	Orbit 3	Orbit 4
γ^\vee	$[0, 0, -1, 0, 0, 0]$	$[-1, 0, 0, 1, 0, 0]$	$[1, -1, 0, 0, 0, 0]$	$[-1, 0, 0, 0, 0, 0]$
$\mathbf{c}^*\gamma^\vee$	$0, 0, 1, 0, -1, 0$	$1, 0, 0, -1, -1, 1$	$-1, 1, 0, 0, 0, -1$	$1, 0, 0, 0, -1, 0$
$(\mathbf{c}^*)^2\gamma^\vee$	$1, 1, 0, 0, -1, -1$	$0, 0, 1, 0, 0, -1$	$1, 0, 0, 1, -1, 0$	$0, 1, 1, 0, -1, -1$
$(\mathbf{c}^*)^3\gamma^\vee$	$0, 1, 1, 1, -1, -1$	$0, 1, -1, 1, 0, -1$	$0, 1, 1, -1, -1, 0$	$1, 1, 0, 1, -1, -1$
$(\mathbf{c}^*)^4\gamma^\vee$	$1, 1, 0, 0, -1, 0$	$0, 0, 1, 0, -1, 1$	$1, 0, 0, 1, 0, -1$	$0, 1, 1, 0, -1, 0$
$(\mathbf{c}^*)^5\gamma^\vee$	$[0, 0, 1, 0, 0, 0]$	$[1, 0, 0, -1, 0, 0]$	$[-1, 1, 0, 0, 0, 0]$	$[1, 0, 0, 0, 0, 0]$
$(\mathbf{c}^*)^6\gamma^\vee$	$0, 0, -1, 0, 1, 0$	$-1, 0, 0, 1, 1, -1$	$1, -1, 0, 0, 0, 1$	$-1, 0, 0, 0, 1, 0$
$(\mathbf{c}^*)^7\gamma^\vee$	$-1, -1, 0, 0, 1, 1$	$0, 0, -1, 0, 0, 1$	$-1, 0, 0, -1, 1, 0$	$0, -1, -1, 0, 1, 1$
$(\mathbf{c}^*)^8\gamma^\vee$	$0, -1, -1, -1, 1, 1$	$0, -1, 1, -1, 0, 1$	$0, -1, -1, 1, 1, 0$	$-1, -1, 0, -1, 1, 1$
$(\mathbf{c}^*)^9\gamma^\vee$	$-1, -1, 0, 0, 1, 0$	$0, 0, -1, 0, 1, -1$	$-1, 0, 0, -1, 0, 1$	$0, -1, -1, 0, 1, 0$
	Orbit 5	Orbit 6	Orbit 7	Orbit 8
γ^\vee	$[0, 0, -1, 1, 0, 0]$	$[0, -1, 1, 0, 0, 0]$	$[0, 0, 0, -1, 0, 0]$	$[1, 0, -1, 0, 0, 0]$
$\mathbf{c}^*\gamma^\vee$	$0, 0, 1, -1, -1, 1$	$0, 1, -1, 0, 0, -1$	$0, 0, 0, 1, 0, -1$	$[-1, 0, 1, 0, 0, 0]$
$(\mathbf{c}^*)^2\gamma^\vee$	$1, 0, 0, 0, 0, -1$	$0, 0, 1, 1, -1, 0$	$0, 1, 0, 0, -1, 0$	
$(\mathbf{c}^*)^3\gamma^\vee$	$-1, 1, 0, 1, 0, -1$	$1, 1, 0, -1, -1, 0$	$1, 0, 1, 0, -1, 0$	
$(\mathbf{c}^*)^4\gamma^\vee$	$1, 0, 0, 0, -1, 1$	$0, 0, 1, 1, 0, -1$	$0, 1, 0, 0, 0, -1$	
$(\mathbf{c}^*)^5\gamma^\vee$	$[0, 0, 1, -1, 0, 0]$	$[0, 1, -1, 0, 0, 0]$	$[0, 0, 0, 1, 0, 0]$	
$(\mathbf{c}^*)^6\gamma^\vee$	$0, 0, -1, 1, 1, -1$	$0, -1, 1, 0, 0, 1$	$0, 0, 0, -1, 0, 1$	
$(\mathbf{c}^*)^7\gamma^\vee$	$-1, 0, 0, 0, 0, 1$	$0, 0, -1, -1, 1, 0$	$0, -1, 0, 0, 1, 0$	
$(\mathbf{c}^*)^8\gamma^\vee$	$1, -1, 0, -1, 0, 1$	$-1, -1, 0, 1, 1, 0$	$-1, 0, -1, 0, 1, 0$	
$(\mathbf{c}^*)^9\gamma^\vee$	$-1, 0, 0, 0, 1, -1$	$0, 0, -1, -1, 0, 1$	$0, -1, 0, 0, 0, 1$	
		Orbit 9	Orbit 10	
γ^\vee		$[0, -1, 1, 1, 0, 0]$	$[1, -1, 0, 1, 0, 0]$	
$\mathbf{c}^*\gamma^\vee$		$[0, 1, -1, -1, 0, 0]$	$[-1, 1, 0, -1, 0, 0]$	

TABLE B.15. \mathbf{D}_6 , there exist 10 semi-Coxeter orbits: 7 of length 10, and three orbit of length 2, all orbits contain unicolored linkage diagrams. All orbits are self-opposite

B.2. Semi-Coxeter orbits for $D_7(a_i)$, $E_7(a_i)$, D_7 , E_7 .

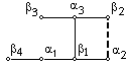
 $E_7(\mathbf{a}_1)$	Orbit 1	Orbit 2	Orbit 3	Orbit 4
γ^\vee	$[0, 0, 0, 0, 1, -1, 0]$	$[0, 0, 0, 0, 0, 0, -1]$	$[0, 0, 0, 0, -1, 0, 0]$	$[0, 0, 0, 0, 0, -1, 1]$
$\mathbf{c}^*\gamma^\vee$	$0, 1, 0, -1, 0, 1, 0$	$1, 0, 0, -1, 0, 0, 0$	$0, -1, 1, 0, -1, -1, 0$	$-1, 0, 1, 0, -1, 0, 0$
$(\mathbf{c}^*)^2\gamma^\vee$	$1, 0, 0, 0, 0, -1, -1$	$0, 1, 1, -1, 0, -1, 0$	$0, 0, 1, -1, 0, 0, 0$	$-1, 1, 0, 0, 0, 0, -1$
$(\mathbf{c}^*)^3\gamma^\vee$	$0, 0, 1, -1, -1, 0, 1$	$1, 0, 1, -1, -1, 0, -1$	$1, 1, 0, -1, 1, 0, -1$	$0, 1, 0, -1, 1, 0, 1$
$(\mathbf{c}^*)^4\gamma^\vee$	$0, 0, 1, 0, 0, -1, -1$	$1, 0, 1, -1, 0, -1, 0$	$1, 1, 0, -1, 0, 0, 0$	$0, 1, 0, 0, 0, 0, -1$
$(\mathbf{c}^*)^5\gamma^\vee$	$1, 0, 0, -1, 0, 1, 0$	$0, 1, 1, -1, 0, 0, 0$	$0, 0, 1, 0, -1, -1, 0$	$1, -1, 0, 0, -1, 0, 0$
$(\mathbf{c}^*)^6\gamma^\vee$	$0, 1, 0, 0, 1, -1, 0$	$1, 0, 0, 0, 0, 0, -1$	$0, -1, 1, 0, -1, 0, 0$	$-1, 0, 1, 0, 0, -1, 1$
$(\mathbf{c}^*)^7\gamma^\vee$	$[0, 0, 0, 0, -1, 1, 0]$	$[0, 0, 0, 0, 0, 0, 1]$	$[0, 0, 0, 0, 1, 0, 0]$	$[0, 0, 0, 0, 0, 1, -1]$
$(\mathbf{c}^*)^8\gamma^\vee$	$0, -1, 0, 1, 0, -1, 0$	$-1, 0, 0, 1, 0, 0, 0$	$0, 1, -1, 0, 1, 1, 0$	$1, 0, -1, 0, 1, 0, 0$
$(\mathbf{c}^*)^9\gamma^\vee$	$-1, 0, 0, 0, 0, 1, 1$	$0, -1, -1, 1, 0, 1, 0$	$0, 0, -1, 1, 0, 0, 0$	$-1, 1, 0, 0, 0, 0, 1$
$(\mathbf{c}^*)^{10}\gamma^\vee$	$0, 0, -1, 1, 1, 0, -1$	$-1, 0, -1, 1, 1, 0, 1$	$-1, -1, 0, 1, -1, 0, 1$	$0, -1, 0, 1, -1, 0, -1$
$(\mathbf{c}^*)^{11}\gamma^\vee$	$0, 0, -1, 0, 0, 1, 1$	$-1, 0, -1, 1, 0, 1, 0$	$-1, -1, 0, 1, 0, 0, 0$	$0, -1, 0, 0, 0, 0, 1$
$(\mathbf{c}^*)^{12}\gamma^\vee$	$-1, 0, 0, 1, 0, -1, 0$	$0, -1, -1, 1, 0, 0, 0$	$0, 0, -1, 0, 1, 1, 0$	$-1, 1, 0, 0, 1, 0, 0$
$(\mathbf{c}^*)^{13}\gamma^\vee$	$0, -1, 0, 0, -1, 1, 0$	$-1, 0, 0, 0, 0, 0, 1$	$0, 1, -1, 0, 1, 0, 0$	$1, 0, -1, 0, 0, 1, -1$

TABLE B.16. $E_7(\mathbf{a}_1)$, there exist 4 semi-Coxeter orbits, each of which of length 14, all orbits are self-opposite

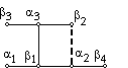
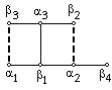
 $E_7(\mathbf{a}_2)$	Orbit 1	Orbit 2	Orbit 3 (orbit 5 is opposite)	Orbit 4 (orbit 6 is opposite) (no unicolored)
γ^\vee	$0, -1, 1, 0, -1, 0, 0$	$[0, 0, 0, 0, 1, -1, 1]$	$[0, 0, 0, 0, 0, -1, 0]$	$-1, 0, 0, 0, 0, 1, 0$
$\mathbf{c}^*\gamma^\vee$	$[0, 0, 0, 0, 1, 0, 0]$	$[0, 0, 0, 0, -1, 1, -1]$	$0, 0, 1, -1, -1, 0, 0$	$1, 0, -1, 0, 1, 0, 0$
$(\mathbf{c}^*)^2\gamma^\vee$	$0, 1, -1, 0, 1, 1, -1$		$1, 0, 1, -1, 0, -1, 0$	$-1, 1, 0, 0, 0, 0, -1$
$(\mathbf{c}^*)^3\gamma^\vee$	$0, 1, -1, 0, 1, 0, 0$		$0, 1, 1, -1, 0, 0, -1$	$1, 0, 0, -1, 0, 0, 1$
$(\mathbf{c}^*)^4\gamma^\vee$	$[0, 0, 0, 0, -1, 0, 0]$		$1, 1, 0, -1, 1, 0, 0$	$0, 0, 1, 0, -1, -1, -1$
$(\mathbf{c}^*)^5\gamma^\vee$	$0, -1, 1, 0, -1, -1, 1$		$0, 1, 0, 0, 0, 0, -1$	$0, 0, 1, -1, 0, 0, 1$
$(\mathbf{c}^*)^6\gamma^\vee$			$[0, 0, 0, 0, 0, 0, 1]$	$1, 0, 0, 0, 0, 0, -1$
$(\mathbf{c}^*)^7\gamma^\vee$			$0, -1, 0, 1, -1, 0, 0$	$-1, 1, 0, 0, 1, 0, 0$
$(\mathbf{c}^*)^8\gamma^\vee$			$-1, -1, 0, 1, 0, 0, 1$	$1, 0, -1, 0, 0, 1, 0$
$(\mathbf{c}^*)^9\gamma^\vee$			$0, -1, -1, 1, 0, 1, 0$	$-1, 0, 0, 1, 0, -1, 0$
$(\mathbf{c}^*)^{10}\gamma^\vee$			$-1, 0, -1, 1, 1, 0, 0$	$0, -1, 0, 0, -1, 1, 1$
$(\mathbf{c}^*)^{11}\gamma^\vee$			$0, 0, -1, 0, 0, 1, 0$	$0, -1, 0, 1, 0, -1, 0$

TABLE B.17. $E_7(\mathbf{a}_2)$, 6 semi-Coxeter orbits. Orbits 1 and 2 are self-opposite. Orbits 3 and 5 (resp. 4 and 6) are opposite

 E₇(a₃)	Orbit 1		Orbit 1 (cont.)
γ^\vee	$[0, 0, 0, 0, 0, 1, 0]$	$(\mathbf{c}^*)^{15}\gamma^\vee$	$[0, 0, 0, 0, 0, -1, 0]$
$\mathbf{c}^*\gamma^\vee$	$1, 0, -1, 0, 1, 1, 0$	$(\mathbf{c}^*)^{16}\gamma^\vee$	$-1, 0, 1, 0, -1, -1, 0$
$(\mathbf{c}^*)^2\gamma^\vee$	$0, 1, -1, 0, 1, 0, -1$	$(\mathbf{c}^*)^{17}\gamma^\vee$	$0, -1, 1, 0, -1, 0, 1$
$(\mathbf{c}^*)^3\gamma^\vee$	$0, 1, 0, -1, 0, 0, 0$	$(\mathbf{c}^*)^{18}\gamma^\vee$	$0, -1, 0, 1, 0, 0, 0$
$(\mathbf{c}^*)^4\gamma^\vee$	$1, 0, 1, -1, -1, 0, 0$	$(\mathbf{c}^*)^{19}\gamma^\vee$	$-1, 0, -1, 1, 1, 0, 0$
$(\mathbf{c}^*)^5\gamma^\vee$	$0, 0, 1, 0, 0, -1, 0$	$(\mathbf{c}^*)^{20}\gamma^\vee$	$0, 0, -1, 0, 0, 1, 0$
$(\mathbf{c}^*)^6\gamma^\vee$	$-1, 0, 0, 1, 0, 0, 0$	$(\mathbf{c}^*)^{21}\gamma^\vee$	$1, 0, 0, -1, 0, 0, 0$
$(\mathbf{c}^*)^7\gamma^\vee$	$0, -1, -1, 1, 0, 1, 1$	$(\mathbf{c}^*)^{22}\gamma^\vee$	$0, 1, 1, -1, 0, -1, -1$
$(\mathbf{c}^*)^8\gamma^\vee$	$0, -1, -1, 1, 0, 0, 0$	$(\mathbf{c}^*)^{23}\gamma^\vee$	$0, 1, 1, -1, 0, 0, 0$
$(\mathbf{c}^*)^9\gamma^\vee$	$-1, 0, 0, 0, 0, -1, 0$	$(\mathbf{c}^*)^{24}\gamma^\vee$	$1, 0, 0, 0, 0, 1, 0$
$(\mathbf{c}^*)^{10}\gamma^\vee$	$0, 0, 1, -1, -1, 0, 0$	$(\mathbf{c}^*)^{25}\gamma^\vee$	$0, 0, -1, 1, 1, 0, 0$
$(\mathbf{c}^*)^{11}\gamma^\vee$	$1, 0, 1, -1, 0, 0, 0$	$(\mathbf{c}^*)^{26}\gamma^\vee$	$-1, 0, -1, 1, 0, 0, 0$
$(\mathbf{c}^*)^{12}\gamma^\vee$	$0, 1, 0, 0, 1, 0, -1$	$(\mathbf{c}^*)^{27}\gamma^\vee$	$0, -1, 0, 0, -1, 0, 1$
$(\mathbf{c}^*)^{13}\gamma^\vee$	$0, 1, -1, 0, 1, 1, 0$	$(\mathbf{c}^*)^{28}\gamma^\vee$	$0, -1, 1, 0, -1, -1, 0$
$(\mathbf{c}^*)^{14}\gamma^\vee$	$1, 0, -1, 0, 0, 1, 0$	$(\mathbf{c}^*)^{29}\gamma^\vee$	$-1, 0, 1, 0, 0, -1, 0$

	Orbit 2	Orbit 3	Orbit 4
γ^\vee	$[0, 0, 0, 0, 0, 0, -1]$	$[0, 0, 0, 0, 1, -1, 0]$	$[0, 0, 0, 0, -1, 0, -1]$
$\mathbf{c}^*\gamma^\vee$	$0, 1, 0, -1, 1, 0, 0$	$-1, 1, 0, 0, 0, 0, -1$	$0, 0, 1, -1, 0, -1, 1$
$(\mathbf{c}^*)^2\gamma^\vee$	$1, 1, 0, -1, 0, 1, -1$	$1, 0, 0, -1, 0, 1, 1$	$0, 0, 1, 0, -1, 0, -1$
$(\mathbf{c}^*)^3\gamma^\vee$	$1, 1, 0, -1, 1, 0, 0$	$1, 0, 0, 0, 0, 0, -1$	$[0, 0, 0, 0, 1, 0, 1]$
$(\mathbf{c}^*)^4\gamma^\vee$	$0, 1, 0, 0, 0, 0, -1$	$-1, 1, 0, 0, 1, -1, 0$	$0, 0, -1, 1, 0, 1, -1$
$(\mathbf{c}^*)^5\gamma^\vee$	$[0, 0, 0, 0, 0, 0, 1]$	$[0, 0, 0, 0, -1, 1, 0]$	$0, 0, -1, 0, 1, 0, 1$
$(\mathbf{c}^*)^6\gamma^\vee$	$0, -1, 0, 1, -1, 0, 0$	$1, -1, 0, 0, 0, 0, 1$	
$(\mathbf{c}^*)^7\gamma^\vee$	$-1, -1, 0, 1, 0, -1, 1$	$-1, 0, 0, 1, 0, -1, -1$	
$(\mathbf{c}^*)^8\gamma^\vee$	$-1, -1, 0, 1, -1, 0, 1$	$-1, 0, 0, 0, 0, 0, 1$	
$(\mathbf{c}^*)^9\gamma^\vee$	$0, -1, 0, 0, 0, 0, 1$	$1, -1, 0, 0, -1, 1, 0$	

TABLE B.18. **E₇(a₃)**, there exist 4 semi-Coxeter orbits, one of length 30, two of length 10 and one of length 6. All orbits are self-opposite

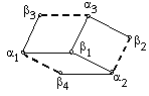
 <p>$E_7(\mathbf{a}_4)$</p>	Orbit 1	Orbit 2	Orbit 3	Orbit 4 (no unicolored)
γ^\vee $\mathbf{c}^*\gamma^\vee$ $(\mathbf{c}^*)^2\gamma^\vee$ $(\mathbf{c}^*)^3\gamma^\vee$ $(\mathbf{c}^*)^4\gamma^\vee$ $(\mathbf{c}^*)^5\gamma^\vee$	<div>0, 0, 0, 0, 0, 1</div> 1, -1, 0, 0, -1, -1, 1 1, -1, 0, 0, 0, 0, 1 <div>0, 0, 0, 0, 0, -1</div> -1, 1, 0, 0, 1, 1, -1 -1, 1, 0, 0, 0, 0, -1	<div>0, 0, 0, 0, 0, -1, 0</div> 1, 0, -1, 0, 1, -1, 1 1, 0, -1, 0, 0, -1, 0 <div>0, 0, 0, 0, 0, 1, 0</div> -1, 0, 1, 0, -1, 1, -1 -1, 0, 1, 0, 0, 1, 0	<div>0, 0, 0, 0, -1, 0, 0</div> 0, -1, 1, 0, -1, 1, 1 0, -1, 1, 0, -1, 0, 0 <div>0, 0, 0, 0, 1, 0, 0</div> 0, 1, -1, 0, 1, -1, -1 0, 1, -1, 0, 1, 0, 0	-1, 0, 0, 0, 0, 1, 0 0, 0, 1, -1, -1, 0, 0 1, 0, 1, -1, 0, 0, 1 1, 0, 0, 0, 0, -1, 0 0, 0, -1, 1, 1, 0, 0 -1, 0, -1, 1, 0, 0, -1
	Orbit 5 (no unicolored)	Orbit 6 (no unicolored)	Orbit 7 (no unicolored)	Orbit 8 (no unicolored)
γ^\vee $\mathbf{c}^*\gamma^\vee$ $(\mathbf{c}^*)^2\gamma^\vee$ $(\mathbf{c}^*)^3\gamma^\vee$ $(\mathbf{c}^*)^4\gamma^\vee$ $(\mathbf{c}^*)^5\gamma^\vee$	0, 0, -1, 0, 0, -1, 0 1, 0, 0, -1, 0, 0, 1 1, 0, 1, -1, -1, 0, 0 0, 0, 1, 0, 0, 1, 0 -1, 0, 0, 1, 0, 0, -1 -1, 0, -1, 1, 1, 0, 0	0, 1, 1, -1, 0, 1, 0 0, 0, 1, 0, -1, 0, 0 0, -1, 0, 1, 0, 0, 1 0, -1, -1, 1, 0, -1, 0 0, 0, -1, 0, 1, 0, 0 0, 1, 0, -1, 0, 0, -1	0, -1, 0, 0, -1, 0, 0 0, 0, 1, -1, 0, 1, 0 0, 1, 1, -1, 0, 0, -1 0, 1, 0, 0, 1, 0, 0 0, 0, -1, 1, 0, -1, 0 0, -1, -1, 1, 0, 0, 1	-1, -1, 0, 1, -1, 0, 0 0, -1, 0, 0, 0, 0, 1 1, 0, 0, -1, 0, -1, 0 1, 1, 0, -1, 1, 0, 0 0, 1, 0, 0, 0, 0, -1 -1, 0, 0, 1, 0, 1, 0
		Orbit 9 (no unicolored)	Orbit 10	
γ^\vee $\mathbf{c}^*\gamma^\vee$ $(\mathbf{c}^*)^2\gamma^\vee$ $(\mathbf{c}^*)^3\gamma^\vee$ $(\mathbf{c}^*)^4\gamma^\vee$ $(\mathbf{c}^*)^5\gamma^\vee$		0, 1, 0, -1, 1, 0, 0 1, 1, 0, -1, 0, -1, 0 1, 0, 0, 0, 0, 0, 1 0, -1, 0, 1, -1, 0, 0 -1, -1, 0, 1, 0, 1, 0 -1, 0, 0, 0, 0, 0, -1	<div>0, 0, 0, 0, 1, 1, 1</div> <div>0, 0, 0, 0, -1, -1, -1</div>	

TABLE B.19. $E_7(\mathbf{a}_4)$, 10 semi-Coxeter orbits, nine of length 6, one of length 2

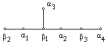
 E₇	Orbit 1	Orbit 2	Orbit 3	Orbit 4
γ^\vee	$[0, 0, 0, -1, 0, 0, 0]$	$[-1, 0, 1, 0, 0, 0, 0]$	$[1, -1, 0, 0, 0, 0, 0]$	$[0, 1, -1, -1, 0, 0, 0]$
$\mathbf{c}^*\gamma^\vee$	$[0, 0, 0, 1, 0, 0, -1]$	$[1, 0, -1, 0, 0, -1, 0]$	$[-1, 1, 0, 0, 0, 1, -1]$	$[0, -1, 1, 1, 0, 0, 0]$
$(\mathbf{c}^*)^2\gamma^\vee$	$[0, 1, 0, 0, -1, 0, 0]$	$[0, 0, 1, 0, -1, 1, 0]$	$[0, 0, 0, 1, 0, -1, 0]$	
$(\mathbf{c}^*)^3\gamma^\vee$	$[1, 0, 1, 0, -1, -1, 0]$	$[0, 1, 0, 0, 0, -1, -1]$	$[1, 0, 0, -1, -1, 0, 1]$	
$(\mathbf{c}^*)^4\gamma^\vee$	$[1, 1, 0, 0, -1, 0, -1]$	$[1, 0, 0, 1, -1, 0, 0]$	$[0, 0, 1, 0, 0, 0, -1]$	
$(\mathbf{c}^*)^5\gamma^\vee$	$[0, 1, 1, 1, -1, 0, -1]$	$[0, 1, 1, -1, -1, 0, 0]$	$[0, 1, -1, 1, 0, 0, -1]$	
$(\mathbf{c}^*)^6\gamma^\vee$	$[1, 1, 0, 0, -1, -1, 0]$	$[1, 0, 0, 1, 0, -1, -1]$	$[0, 0, 1, 0, -1, 0, 1]$	
$(\mathbf{c}^*)^7\gamma^\vee$	$[1, 0, 1, 0, -1, 0, 0]$	$[0, 1, 0, 0, -1, 1, 0]$	$[1, 0, 0, -1, 0, -1, 0]$	
$(\mathbf{c}^*)^8\gamma^\vee$	$[0, 1, 0, 0, 0, 0, -1]$	$[0, 0, 1, 0, 0, -1, 0]$	$[0, 0, 0, 1, 0, 1, -1]$	
$(\mathbf{c}^*)^9\gamma^\vee$	$[0, 0, 0, 1, 0, 0, 0]$	$[1, 0, -1, 0, 0, 0, 0]$	$[-1, 1, 0, 0, 0, 0, 0]$	
$(\mathbf{c}^*)^{10}\gamma^\vee$	$[0, 0, 0, -1, 0, 0, 1]$	$[-1, 0, 1, 0, 0, 1, 0]$	$[1, -1, 0, 0, 0, -1, 1]$	
$(\mathbf{c}^*)^{11}\gamma^\vee$	$[0, -1, 0, 0, 1, 0, 0]$	$[0, 0, -1, 0, 1, -1, 0]$	$[0, 0, 0, -1, 0, 1, 0]$	
$(\mathbf{c}^*)^{12}\gamma^\vee$	$[-1, 0, -1, 0, 1, 1, 0]$	$[0, -1, 0, 0, 0, 1, 1]$	$[-1, 0, 0, 1, 1, 0, -1]$	
$(\mathbf{c}^*)^{13}\gamma^\vee$	$[-1, -1, 0, 0, 1, 0, 1]$	$[-1, 0, 0, -1, 1, 0, 0]$	$[0, 0, -1, 0, 0, 0, 1]$	
$(\mathbf{c}^*)^{14}\gamma^\vee$	$[0, -1, -1, -1, 1, 0, 1]$	$[0, -1, -1, 1, 1, 0, 0]$	$[0, -1, 1, -1, 0, 0, 1]$	
$(\mathbf{c}^*)^{15}\gamma^\vee$	$[-1, -1, 0, 0, 1, 1, 0]$	$[-1, 0, 0, -1, 0, 1, 1]$	$[0, 0, -1, 0, 1, 0, -1]$	
$(\mathbf{c}^*)^{16}\gamma^\vee$	$[-1, 0, -1, 0, 1, 0, 0]$	$[0, -1, 0, 0, 1, -1, 0]$	$[-1, 0, 0, 1, 0, 1, 0]$	
$(\mathbf{c}^*)^{17}\gamma^\vee$	$[0, -1, 0, 0, 0, 0, 1]$	$[0, 0, -1, 0, 0, 1, 0]$	$[0, 0, 0, -1, 0, -1, 1]$	

TABLE B.20. **E₇**, 4 semi-Coxeter orbits, three of length 18 and one of length 2. All orbits are self-opposite. All orbits contain β -unicolored linkage diagrams

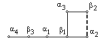
 $D_7(\mathbf{a}_1)$	Orbit 1	Orbit 2	Orbit 3	Orbit 4
γ^\vee	$[0, 0, 1, 0, 0, 0, 0]$	$[1, -1, 0, 0, 0, 0, 0]$	$[0, -1, 0, 1, 0, 0, 0]$	$[-1, 0, 1, 1, 0, 0, 0]$
$\mathbf{c}^*\gamma^\vee$	$0, 0, -1, 0, 1, 1, 0$	$-1, 1, 0, 0, 0, 1, 1$	$0, 1, 0, -1, -1, 1, 1$	$1, 0, -1, -1, 0, 1, 0$
$(\mathbf{c}^*)^2\gamma^\vee$	$-1, 0, -1, 0, 1, 0, 1$	$0, 0, -1, -1, 1, 0, 0$	$0, 1, 0, 0, 0, 0, -1$	$[-1, 1, 0, 1, 0, 0, 0]$
$(\mathbf{c}^*)^3\gamma^\vee$	$-1, -1, 0, -1, 1, -1, 1$	$-1, -1, 0, 1, 1, -1, 0$	$1, -1, 0, 1, 0, -1, -1$	$1, -1, 0, -1, 0, -1, 0$
$(\mathbf{c}^*)^4\gamma^\vee$	$-1, -1, 0, 0, 1, 0, 0$	$0, -1, 0, -1, 0, 0, 1$	$0, 0, 1, 0, -1, 0, 1$	
$(\mathbf{c}^*)^5\gamma^\vee$	$0, 0, -1, 0, 0, 1, 0$	$-1, 1, 0, 0, 0, 1, 0$	$0, 1, 0, -1, 0, 1, 0$	
$(\mathbf{c}^*)^6\gamma^\vee$	$0, 1, 0, 0, -1, 0, 0$	$1, 0, -1, 0, 0, 0, -1$	$0, 0, -1, 1, 1, 0, -1$	
$(\mathbf{c}^*)^7\gamma^\vee$	$1, 0, 1, 0, -1, -1, -1$	$0, 0, 1, 1, -1, -1, 0$	$0, -1, 0, 0, 0, -1, 1$	
$(\mathbf{c}^*)^8\gamma^\vee$	$1, 0, 1, 1, -1, 0, -1$	$1, 0, 1, -1, -1, 0, 0$	$-1, 0, 1, -1, 0, 0, 1$	
$(\mathbf{c}^*)^9\gamma^\vee$	$1, 1, 0, 0, -1, 1, 0$	$0, 1, 0, 1, 0, 1, -1$	$0, 0, -1, 0, 1, 1, -1$	
$(\mathbf{c}^*)^{10}\gamma^\vee$	$[0, 1, 0, 0, 0, 0, 0]$	$[1, 0, -1, 0, 0, 0, 0]$	$[0, 0, -1, 1, 0, 0, 0]$	
$(\mathbf{c}^*)^{11}\gamma^\vee$	$0, -1, 0, 0, 1, -1, 0$	$-1, 0, 1, 0, 0, -1, 1$	$0, 0, 1, -1, -1, -1, 1$	
$(\mathbf{c}^*)^{12}\gamma^\vee$	$-1, -1, 0, 0, 1, 0, 1$	$0, -1, 0, -1, 1, 0, 0$	$0, 0, 1, 0, 0, 0, -1$	
$(\mathbf{c}^*)^{13}\gamma^\vee$	$-1, 0, -1, -1, 1, 1, 1$	$-1, 0, -1, 1, 1, 1, 0$	$1, 0, -1, 1, 0, 1, -1$	
$(\mathbf{c}^*)^{14}\gamma^\vee$	$-1, 0, -1, 0, 1, 0, 0$	$0, 0, -1, -1, 0, 0, 1$	$0, 1, 0, 0, -1, 0, 1$	
$(\mathbf{c}^*)^{15}\gamma^\vee$	$0, -1, 0, 0, 0, -1, 0$	$-1, 0, 1, 0, 0, -1, 0$	$0, 0, 1, -1, 0, -1, 0$	
$(\mathbf{c}^*)^{16}\gamma^\vee$	$0, 0, 1, 0, -1, 0, 0$	$1, -1, 0, 0, 0, 0, -1$	$0, -1, 0, 1, 1, 0, -1$	
$(\mathbf{c}^*)^{17}\gamma^\vee$	$1, 1, 0, 0, -1, 1, -1$	$0, 1, 0, 1, -1, 1, 0$	$0, 0, -1, 0, 0, 1, 1$	
$(\mathbf{c}^*)^{18}\gamma^\vee$	$1, 1, 0, 1, -1, 0, -1$	$1, 1, 0, -1, -1, 0, 0$	$-1, 1, 0, -1, 0, 0, 1$	
$(\mathbf{c}^*)^{19}\gamma^\vee$	$1, 0, 1, 0, -1, -1, 0$	$0, 0, 1, 1, 0, -1, -1$	$0, -1, 0, 0, 1, -1, -1$	
		Orbit 5	Orbit 6	
γ^\vee		$[0, 0, 0, -1, 0, 0, 0]$	$[0, 0, 0, 0, 0, -1, 0]$	
$\mathbf{c}^*\gamma^\vee$		$0, 0, 0, 1, 0, 0, -1$	$0, -1, 1, 0, 0, -1, 0$	
$(\mathbf{c}^*)^2\gamma^\vee$		$1, 0, 0, 0, -1, 0, 0$	$[0, 0, 0, 0, 0, 1, 0]$	
$(\mathbf{c}^*)^3\gamma^\vee$		$0, 1, 1, 0, -1, 0, 0$	$0, 1, -1, 0, 0, 1, 0$	
$(\mathbf{c}^*)^4\gamma^\vee$		$1, 0, 0, 0, 0, 0, -1$		
$(\mathbf{c}^*)^5\gamma^\vee$		$[0, 0, 0, 1, 0, 0, 0]$		
$(\mathbf{c}^*)^6\gamma^\vee$		$0, 0, 0, -1, 0, 0, 1$		
$(\mathbf{c}^*)^7\gamma^\vee$		$-1, 0, 0, 0, 1, 0, 0$		
$(\mathbf{c}^*)^8\gamma^\vee$		$0, -1, -1, 0, 1, 0, 0$		
$(\mathbf{c}^*)^9\gamma^\vee$		$-1, 0, 0, 0, 0, 0, 1$		

TABLE B.21. $D_7(\mathbf{a}_1)$, there exist 10 semi-Coxeter orbits. Semi-Coxeter orbits 1-4 (three of length 20 and one of length 4) belong to the first E -type component. Every orbit out of 1-4 has the opposite one (7-10) lying in the second E -type component. Orbits 5 and 6 are self-opposite. All orbits contain α -unicolored or β -unicolored linkage diagrams


 $\mathbf{D}_7(\mathbf{a}_2)$	Orbit 1	Orbit 2	Orbit 3	Orbit 4
γ^\vee	$[0, 0, 1, 0, 0, 0, 0]$	$[0, 1, 0, 1, 0, 0, 0]$	$[1, -1, 0, 0, 0, 0, 0]$	$[1, 0, -1, 1, 0, 0, 0]$
$\mathbf{c}^*\gamma^\vee$	$0, 0, -1, 0, 1, 1, 0$	$0, -1, 0, -1, 1, 0, 0$	$-1, 1, 0, 0, 0, 1, 1$	$-1, 0, 1, -1, 0, 0, 1$
$(\mathbf{c}^*)^2\gamma^\vee$	$-1, 0, -1, -1, 1, 1, 1$	$-1, 0, -1, 1, 1, 0, 1$	$0, 0, -1, -1, 1, 1, -1$	$0, 0, -1, 1, 1, 0, -1$
$(\mathbf{c}^*)^3\gamma^\vee$	$-1, 0, -1, 0, 1, 0, 0$	$-1, -1, 0, -1, 1, 0, 0$	$0, 0, -1, 0, 0, 0, 1$	$0, -1, 0, -1, 0, 0, 1$
$(\mathbf{c}^*)^4\gamma^\vee$	$0, -1, 0, 0, 0, -1, 0$	$[0, 0, -1, 1, 0, 0, 0]$	$-1, 0, 1, 0, 0, -1, 0$	$[-1, 1, 0, 1, 0, 0, 0]$
$(\mathbf{c}^*)^5\gamma^\vee$	$0, 0, 1, 1, -1, -1, 0$	$0, 0, 1, -1, -1, 0, 0$	$1, -1, 0, 1, 0, -1, -1$	$1, -1, 0, -1, 0, 0, -1$
$(\mathbf{c}^*)^6\gamma^\vee$	$1, 0, 1, 0, -1, 0, -1$	$1, 1, 0, 1, -1, 0, -1$	$0, 0, 1, 0, -1, 0, 1$	$0, 1, 0, 1, -1, 0, 1$
$(\mathbf{c}^*)^7\gamma^\vee$	$1, 1, 0, 0, -1, 1, 0$	$1, 0, 1, -1, -1, 0, 0$	$0, 1, 0, 0, 0, 1, -1$	$0, 0, 1, -1, 0, 0, -1$
$(\mathbf{c}^*)^8\gamma^\vee$	$0, 1, 0, -1, 0, 1, 0$		$1, 0, -1, -1, 0, 1, 0$	
$(\mathbf{c}^*)^9\gamma^\vee$	$0, 0, -1, 0, 1, 0, 0$		$-1, 1, 0, 0, 0, 0, 1$	
$(\mathbf{c}^*)^{10}\gamma^\vee$	$-1, -1, 0, 0, 1, -1, 1$		$0, -1, 0, 0, 1, -1, -1$	
$(\mathbf{c}^*)^{11}\gamma^\vee$	$-1, -1, 0, 1, 1, -1, 0$		$0, -1, 0, 1, 0, -1, 1$	
$(\mathbf{c}^*)^{13}\gamma^\vee$	$[0, -1, 0, 0, 0, 0, 0]$		$[-1, 0, 1, 0, 0, 0, 0]$	
$(\mathbf{c}^*)^{14}\gamma^\vee$	$0, 1, 0, 0, -1, 1, 0$		$1, 0, -1, 0, 0, 1, -1$	
$(\mathbf{c}^*)^{15}\gamma^\vee$	$1, 1, 0, -1, -1, 1, -1$		$0, 1, 0, -1, -1, 1, 1$	
$(\mathbf{c}^*)^{16}\gamma^\vee$	$1, 1, 0, 0, -1, 0, 0$		$0, 1, 0, 0, 0, 0, -1$	
$(\mathbf{c}^*)^{17}\gamma^\vee$	$0, 0, 1, 0, 0, -1, 0$		$1, -1, 0, 0, 0, -1, 0$	
$(\mathbf{c}^*)^{18}\gamma^\vee$	$0, -1, 0, 1, 1, -1, 0$		$-1, 0, 1, 1, 0, -1, 1$	
$(\mathbf{c}^*)^{19}\gamma^\vee$	$-1, -1, 0, 0, 1, 0, 1$		$0, -1, 0, 0, 1, 0, -1$	
$(\mathbf{c}^*)^{20}\gamma^\vee$	$-1, 0, -1, 0, 1, 1, 0$		$0, 0, -1, 0, 0, 1, 1$	
$(\mathbf{c}^*)^{21}\gamma^\vee$	$0, 0, -1, -1, 0, 1, 0$		$-1, 1, 0, -1, 0, 1, 0$	
$(\mathbf{c}^*)^{22}\gamma^\vee$	$0, 1, 0, 0, -1, 0, 0$		$1, 0, -1, 0, 0, 0, -1$	
$(\mathbf{c}^*)^{23}\gamma^\vee$	$1, 0, 1, 0, -1, -1, -1$		$0, 0, 1, 0, -1, -1, 1$	
	$1, 0, 1, 1, -1, -1, 0$		$0, 0, 1, 1, 0, -1, -1$	
		Orbit 5	Orbit 6	
γ^\vee		$[0, 0, 0, -1, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 1]$	
$\mathbf{c}^*\gamma^\vee$		$0, 0, 0, 1, 0, -1, 0$	$-1, 0, 0, 0, 1, 0, 0$	
$(\mathbf{c}^*)^2\gamma^\vee$		$0, -1, 1, 0, 0, -1, 0$	$0, -1, -1, 0, 1, 0, 0$	
$(\mathbf{c}^*)^3\gamma^\vee$		$[0, 0, 0, 1, 0, 0, 0]$	$-1, 0, 0, 0, 0, 0, 1$	
$(\mathbf{c}^*)^4\gamma^\vee$		$0, 0, 0, -1, 0, 1, 0$	$[0, 0, 0, 0, 0, 0, -1]$	
$(\mathbf{c}^*)^5\gamma^\vee$		$0, 1, -1, 0, 0, 1, 0$	$1, 0, 0, 0, -1, 0, 0$	
$(\mathbf{c}^*)^6\gamma^\vee$			$0, 1, 1, 0, -1, 0, 0$	
$(\mathbf{c}^*)^7\gamma^\vee$			$1, 0, 0, 0, 0, 0, -1$	

TABLE B.22. $\mathbf{D}_7(\mathbf{a}_2)$, there exist 10 semi-Coxeter orbits. Semi-Coxeter orbits 1-4 (two of length 24 and two of length 8) belong to the first E -type component. Every orbit out of 1-4 has the opposite one (7-10) lying in the second E -type component. Orbits 5 and 6 are self-opposite. All orbits contain α -unicolored or β -unicolored linkage diagrams

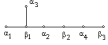
 D₇	Orbit 1 (no unicolored)	Orbit 2	Orbit 3 (no unicolored)	Orbit 4
γ^\vee	0, 0, -1, 0, 0, 1, 0	0, -1, -1, -1, 1, 1, 1	-1, 0, 0, -1, 0, 1, 1	0, -1, 1, 1, 0, 0, 0
$\mathbf{c}^*\gamma^\vee$	0, -1, 1, -1, 0, 1, 1	-1, -1, 0, -1, 1, 1, 0	1, -1, 0, -1, 0, 1, 0	0, 1, -1, -1, 0, 0, 1
$(\mathbf{c}^*)^2\gamma^\vee$	0, 0, -1, -1, 1, 0, 0	0, -1, -1, 0, 1, 0, 0	-1, 0, 0, 0, 1, -1, 0	0, -1, 1, 0, 0, 1, -1
$(\mathbf{c}^*)^3\gamma^\vee$	-1, -1, 0, 1, 1, 0, -1	-1, 0, 0, 0, 0, 0, 0	0, 0, -1, 1, 0, 0, -1	0, 0, -1, 0, 1, -1, 1
$(\mathbf{c}^*)^4\gamma^\vee$	0, 0, -1, 0, 0, 0, 1	1, 0, 0, 0, -1, 0, 0	0, 0, 1, 0, -1, 0, 1	-1, 0, 0, 0, 0, 1, -1
$(\mathbf{c}^*)^5\gamma^\vee$	0, 0, 1, -1, -1, 1, 0	0, 1, 1, 0, -1, -1, 0	1, 1, 0, -1, -1, 0, 0	1, -1, 0, -0, 0, 0, 1
$(\mathbf{c}^*)^6\gamma^\vee$	1, 0, 0, 0, 0, -1, 0	1, 1, 0, 1, -1, -1, -1	0, 0, 1, 1, 0, -1, -1	-1, 1, 0, -1, 0, 0, 0
$(\mathbf{c}^*)^7\gamma^\vee$	-1, 1, 0, 1, 0, -1, -1	0, 1, 1, 1, -1, -1, 0	0, 1, -1, 1, 0, -1, 0	1, -1, 0, 1, 0, 0, -1
$(\mathbf{c}^*)^8\gamma^\vee$	1, 0, 0, 1, -1, 0, 0	1, 1, 0, 0, -1, 0, 0	0, 0, 1, 0, -1, 1, 0	-1, 1, 0, 0, 0, -1, 1
$(\mathbf{c}^*)^9\gamma^\vee$	0, 1, 1, -1, -1, 0, 1	0, 0, 1, 0, 0, 0, 0	1, 0, 0, -1, 0, 0, 1	1, 0, 0, 0, -1, 1, -1
$(\mathbf{c}^*)^{10}\gamma^\vee$	1, 0, 0, 0, 0, 0, -1	0, 0, -1, 0, 1, 0, 0	-1, 0, 0, 0, 1, 0, -1	0, 0, 1, 0, 0, -1, 1
$(\mathbf{c}^*)^{11}\gamma^\vee$	-1, 0, 0, 1, 1, -1, 0	-1, -1, 0, 0, 1, 1, 0	0, -1, -1, 1, 1, 0, 0	0, 1, -1, 0, 0, 0, -1
	Orbit 5	Orbit 6	Orbit 7	Orbit 8
γ^\vee	-1, 1, 0, 0, 0, 0, 0	-1, 0, 0, 1, 0, 0, 0	0, 0, 0, 0, 0, 0, -1	1, 0, -1, 0, 0, 0, 0
$\mathbf{c}^*\gamma^\vee$	1, -1, 0, 0, 0, 1, 0	1, 0, 0, -1, -1, 1, 1	0, 0, 0, 1, 0, -1, 0	-1, 0, 1, 0, 0, 0, 0
$(\mathbf{c}^*)^2\gamma^\vee$	-1, 0, 0, -1, 1, 0, 1	0, 0, 1, -1, 0, 0, 0	0, 1, 0, 0, -1, 0, 0	
$(\mathbf{c}^*)^3\gamma^\vee$	0, -1, -1, 0, 1, 1, -1	0, 0, -1, 1, 1, -1, -1	1, 0, 1, 0, -1, 0, 0	
$(\mathbf{c}^*)^4\gamma^\vee$	-1, -1, 0, 0, 1, 0, 1		0, 1, 0, 0, 0, -1, 0	
$(\mathbf{c}^*)^5\gamma^\vee$	0, 0, -1, -1, 0, 1, 0		0, 0, 0, 1, 0, 0, -1	
$(\mathbf{c}^*)^6\gamma^\vee$	0, -1, 1, 0, 0, 0, 0		0, 0, 0, 0, 0, 0, 1	
$(\mathbf{c}^*)^7\gamma^\vee$	0, 1, -1, 0, 0, -1, 0		0, 0, 0, -1, 0, 1, 0	
$(\mathbf{c}^*)^8\gamma^\vee$	0, 0, 1, 1, -1, 0, -1		0, -1, 0, 0, 1, 0, 0	
$(\mathbf{c}^*)^9\gamma^\vee$	1, 1, 0, 0, -1, -1, 1		-1, 0, -1, 0, 1, 0, 0	
$(\mathbf{c}^*)^{10}\gamma^\vee$	0, 1, 1, 0, -1, 0, -1		0, -1, 0, 0, 0, 1, 0	
$(\mathbf{c}^*)^{11}\gamma^\vee$	1, 0, 0, 1, 0, -1, 0		0, 0, 0, -1, 0, 0, 1	

TABLE B.23. **D₇**, there exist 14 semi-Coxeter orbits. Semi-Coxeter orbits 1-6 (5 of length 12 and one of length 4 belong to the first *E*-type component. Every orbit out of 1-6 has the opposite one (9-14) lying in the second *E*-type component. Orbits 7 and 8 are self-opposite. Orbits 1, 3 (and opposite to these orbits, i.e., 9, 11) do not contain unicolored linkage diagrams

REFERENCES

- [Bo02] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4,5,6*. Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. xii+300 pp.
- [Bo05] N. Bourbaki, *Lie groups and Lie algebras. Chapters 7,8,9*. Translated from the 1975 and 1982 French originals by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. xii+434 pp.
- [Ch84] R. N. Cahn, *Semi-Simple Lie Algebras and Their Representations*. Berkeley, Benjamin-Cummings publishing company, 1984.
- [Ca70] R. W. Carter, *Conjugacy classes in the Weyl group*. 1970 Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69) pp. 297–318 Springer, Berlin.
- [Ca72] R. W. Carter, *Conjugacy classes in the Weyl group*. *Compositio Math.* 25 (1972), 1–59.
- [CE72] R. W. Carter, G. B. Elkington, *A note on the parametrization of conjugacy classes*. *J. Algebra* 20 (1972) 350–354.
- [Dy46] E. B. Dynkin, *Classification of the simple Lie groups*. (Russian) *Rec. Math. [Mat. Sbornik]* N. S. 18(60), (1946). 347–352.
- [Dy00] E. B. Dynkin, *Selected papers of E. B. Dynkin with commentary*. Edited by A. A. Yushkevich, G. M. Seitz and A. L. Onishchik. American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 2000. xxviii+796 pp. ISBN: 0-8218-1065-0.
- [GP00] M. Geck, G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*. London Mathematical Society Monographs. New Series, 21. The Clarendon Press, Oxford University Press, New York, 2000. xvi+446 pp. ISBN: 0-19-850250-8
- [Kac80] V. Kac, *Infnite root systems, representations of graphs and invariant theory*. *Invent. Math.* 56 (1980), no. 1, 57–92.
- [KP85] V. Kac, D. Peterson, *112 constructions of the basic representation of the loop group of E_8* . Symposium on anomalies, geometry, topology (Chicago, Ill., 1985), 276–298, World Sci. Publishing, Singapore, 1985.
- [KOV95] F. I. Karpelevich, A. L. Onishchik, and E. B. Vinberg, *On the work of E. B. Dynkin in the theory of Lie groups in: Lie Groups and Lie Algebras: E. B. Dynkin's Seminar*, 1–12, Amer. Math. Soc. Transl. Ser. 2, 169, Amer. Math. Soc. 1995, 202 pp, vol. 169.
- [Sl81] R. Slansky, *Group theory for unified model building*. *Phys. Rep.* 79 (1981), no. 1, 1–128.
- [Sp74] T. A. Springer, *Regular elements of finite reflection groups*. *Invent. Math.* 25 (1974), 159–198.
- [St08] R. Stekolshchik, *Notes on Coxeter Transformations and the McKay Correspondence*, Springer Monographs in Mathematics, 2008, XX, 240 p.
- [St10.I] R. Stekolshchik, *Root systems and diagram calculus. I. Cycles in the Carter diagrams*, arXiv:1005.2769v4.
- [St10.II] R. Stekolshchik, *Root systems and diagram calculus. II. Quadratic forms for the Carter diagrams*, arXiv:1010.5684v5.
- [St11] R. Stekolshchik, *Root systems and diagram calculus. IV. Linkage systems for the Carter diagrams A_l and B_l* , (in preparation).